Exercises with a $\star$ are eligible for bonus points.

3.1. Complex line integrals

(a) Compute $\int_\gamma \cos(\Re(z))\,dz$, when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

SOL: By setting $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$ we get

$$\begin{aligned}\int_\gamma \cos(\Re(z))\,dz &= \int_0^{2\pi} \cos(\cos(t))(-\sin(t) + i\cos(t))\,dt \\ &= \int_0^{2\pi} (\cos(\cos(t)))'\,dt + i\int_0^{2\pi} \cos(\cos(t))\cos(t)\,dt \\ &= i\int_{-\pi/2}^{\pi/2} \cos(\cos(t))\cos(t)\,dt + i\int_{\pi/2}^{3\pi/2} \cos(\cos(t))\cos(t)\,dt = 0.\end{aligned}$$

(b) Compute $\int_\gamma (\bar{z})^k\,dz$ for any $k \in \mathbb{Z}$ and when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

SOL: Notice that on the unit circle $\bar{z}$ is equal to z^{-1} because $z\bar{z} = |z|^2 = 1$. Hence, when $k \neq 0$, we have that

$$\int_\gamma (\bar{z})^k\,dz = \int_\gamma z^{-k}\,dz = 0,$$

since z^{-k} admits the primitive $z^{-k+1}/(-k+1)$, and γ is in particular a closed curve. When $k = 1$ then we have that

$$\int_\gamma \bar{z}\,dz = \int_\gamma z^{-1}\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = 2\pi i.$$

(c) Compute $\int_\gamma (z^{2023} + \pi z^{11} + i)\,dz$, when γ is the spiral $\{1 + te^{i\pi t} : t \in [0,1]\}$.

SOL: The argument is a polynomial expression, and therefore we can easily find a primitive

$$F(z) = \frac{z^{2024}}{2024} + \frac{\pi z^{12}}{12} + iz.$$

Hence, the integral over γ depends only on its end points:

$$\begin{aligned}\int_\gamma (z^{2023} + \pi z^{11} + i)\,dz = \int_\gamma F'\,dz &= F(\gamma(1)) - F(\gamma(0)) = F(0) - F(1) \\ &= -F(1) = -\frac{1}{2024} - \frac{\pi}{12} + i.\end{aligned}$$

3.2. A polynomial identity Let γ be the counter-clockwise oriented circle of radius $r > 0$ and center $z_0 \in \mathbb{C}$, and let p be any complex polynomial. Show that

$$\int_\gamma p(\bar{z})\, dz = 2\pi i r^2 p'(\bar{z}_0).$$

SOL: Since every complex polynomial can be expressed as a finite $\mathbb{C}$-linear combination of polynomial in the form[1] $\{p_k(z) = (z - \bar{z}_0)^k : k \geq 0\}$, by linearity of the expression we are asked to prove, it is sufficient to show the identity replacing $p(\bar{z})$ with an arbitrary $p_k(\bar{z})$. Notice that $p_0'(\bar{z}_0) = 1$ and $p_k'(\bar{z}_0) = 0$ if $k \neq 0$. We then compute

$$\int_\gamma p_k(\bar{z})\, dx = \int_0^{2\pi} p_k(\overline{z_0 + re^{it}}) ire^{it}\, dt = ir^{k+1} \int_0^{2\pi} (e^{-it})^k e^{it}\, dt,$$

which by Exercise 3.1.(b) it is equal to $2\pi i r^2$ if $k = 1$ and zero otherwise, proving the formula for the basis $\{p_k(z) = (z - \bar{z})^k : k \geq 0\}$, and hence for all complex polynomials by linearity.

3.3. Real integrals via complex integration For the first point you can use Cauchy Theorem (f holomorphic and γ closed implies $\int_\gamma f\, dz = 0$). Also, it could be useful to recall the Gaussian integral $\int_{-\infty}^{+\infty} e^{-t^2}\, dt = \sqrt{\pi}$.

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be the holomorphic function defined by $f(z) = e^{iz^2}$ and $R > 0$. By integrating f over the boundary of $\Omega = \{re^{i\theta} : r \in (0, R), \theta \in (0, \pi/4)\}$, deduce the value of the *Fresnel integrals*

$$\int_0^{+\infty} \cos(x^2)\, dx, \quad \int_0^{+\infty} \sin(x^2)\, dx.$$

SOL: First of all, notice that for all $z = x + iy$ one has that $e^{iz^2} = e^{-2xy}(\cos(x^2 - y^2) + i\sin(x^2 - y^2))$. By Cauchy Theorem we know that since $f(z) = e^{iz^2}$ is holomorphic $\int_{\partial\Omega} f\, dz = 0$. Now, dividing the contour $\partial\Omega$ in elementary curves, we get that

$$0 = \int_{\partial\Omega} f\, dz = \int_0^R f(x)\, dx + \int_0^{\pi/4} f(Re^{it}) iRe^{it}\, dt - \int_0^{R/\sqrt{2}} f(t + it)(1 + i)\, dt,$$

obtainig

$$\int_0^R \cos(x^2) + i\sin(x^2)\, dx = \int_0^{R/\sqrt{2}} f(t + it)(1 + i)\, dt - \int_0^{\pi/4} f(Re^{it}) iRe^{it}\, dt.$$

[1] If you are not convinced, you can prove this elementary fact by induction over the degree of the polynomial for instance.

The Fresnel integrals are equal to the real and respectively imaginary part of the right hand side of the above expression letting $R \to +\infty$. Taking advantage of the Gaussian integral one can compute

$$\lim_{R\to+\infty}\int_0^{R/\sqrt{2}} f(t+it)(1+i)\,dt = \lim_{R\to+\infty}\int_0^{R/\sqrt{2}} e^{-2t^2}(1+i)\,dt = (1+i)\frac{1}{2}\sqrt{\frac{\pi}{2}} = (1+i)\frac{\sqrt{2\pi}}{4}.$$

On the other side, we prove that the remaining integral tends to zero as $R \to +\infty$. In fact, noticing that

$$\cos(t)\sin(t) \geq t/2, \qquad \text{for all } t \in [0, \pi/4],$$

one can estimate

$$\begin{aligned}\left|\int_0^{\pi/4} f(Re^{it})iRe^{it}\,dt\right| &= R\left|\int_0^{\pi/4} e^{it}e^{-2R^2\cos(t)\sin(t)}e^{iR\sin(t)}\,dt\right| \\ &\leq R\int_0^{\pi/4} e^{-2R^2\cos(t)\sin(t)}\,dt = R\int_0^{\pi/4} e^{-R^2 t}\,dt \\ &= -R(e^{-(\pi/4)R^2} - 1)/R^2 \to 0,\end{aligned}$$

as $R \to +\infty$. To summarize

$$\lim_{R\to+\infty}\int_0^R \cos(x^2) + i\sin(x^2)\,dx = (1+i)\frac{\sqrt{2\pi}}{4},$$

proving that

$$\int_0^{+\infty}\cos(x^2)\,dx = \int_0^{+\infty}\sin(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

(b) Let γ be the counter clockwise oriented unit circle and $n \in \mathbb{N}$. Compute

$$\int_\gamma z^{-1}(z+z^{-1})^{2n}\,dz,$$

and deduce that

$$\int_0^{2\pi}\cos(t)^{2n}\,dt = \frac{1}{2^{2n-1}}\binom{2n}{n}\pi.$$

SOL: By Newton binomial expansion we have that

$$\int_\gamma z^{-1}(z+z^{-1})^{2n}\,dz = \sum_{k=0}^{2n}\binom{2n}{k}\int_\gamma z^{-1+k-(2n-k)}\,dz = \sum_{k=0}^{2n}\binom{2n}{k}\int_\gamma z^{2k-2n-1}\,dz.$$

By exercise 3.1 (b) we know that the only term that is different form zero is when $2k-2n-1=-1$, that is when $k=n$, showing

$$\int_\gamma z^{-1}(z+z^{-1})^{2n}\,dz = \binom{2n}{n}2\pi i.$$

On the other side

$$\int_\gamma z^{-1}(z+z^{-1})^{2n}\,dz = \int_0^{2\pi} ie^{it}e^{-it}(e^{it}+e^{-it})^{2n}\,dt = i\int_0^{2\pi}\cos(t)^{2n}2^{2n}\,dt,$$

recalling $\cos(t)=(e^{it}+e^{-it})/2$. Hence

$$\int_0^{2\pi}\cos(t)^{2n}\,dt = \binom{2n}{n}2^{1-2n}\pi,$$

as wished.

3.4. (Challenging and optional) Approximation by polygonal curves Let $\gamma:[a,b]\to\mathbb{C}$ be a closed curve in $\mathbb{C}$ and suppose that there exists a sequence $\gamma_n:[a,b]\to\mathbb{C}$ of polygonal curves, i.e. curves that are piecewise affine, such that $\gamma_n\to\gamma$ and $\gamma_n'\to\gamma'$ uniformly as $n\to+\infty$ (that is $\gamma_n\to\gamma$ with respect to the usual C^1-topology).

(a) Show that any closed C^2-curve γ admit such approximation.

SOL: Consider first a short piece of curve: let $\varepsilon>0$ and $\gamma:[0,\varepsilon]\to\mathbb{C}$ be a C^2-curve. Call $x(s)=\Re(\gamma(s))$ and $y(s)=\Im(\gamma(s))$ the real and imaginary parts of γ, and denote with $\eta:[0,\varepsilon]\to\mathbb{C}$ the linear interpolation between the end points of γ, that is

$$\eta(s)=\gamma(0)+\frac{s}{\varepsilon}(\gamma(\varepsilon)-\gamma(0)),\quad s\in[0,\varepsilon].$$

Set now $M/2 := \max\{|\gamma'(s)|+|\gamma''(s)| : s\in[0,\varepsilon]\}$. Then, since $x(s)$ and $y(s)$ are C^1-functions, by the Mean Value Theorem for every $0\le s_1<s_2\le\varepsilon$ there exist $\xi,\zeta\in[s_1,s_2]$ such that

$$x(s_2)-x(s_1)=(s_2-s_1)x'(\xi),\qquad y(s_2)-y(s_1)=(s_2-s_1)y'(\zeta).$$

In particular

$$|\gamma(s_2)-\gamma(s_1)|^2=(x(s_2)-x(s_1))^2+(y(s_2)-y(s_1))^2=(s_2-s_1)^2(x'(\xi)^2+y'(\zeta)^2)\le M^2\varepsilon^2,$$

implying $|\gamma(s_2)-\gamma(s_1)|/\varepsilon\le M$ for every $0\le s_1<s_2\le\varepsilon$. This allows us to estimate

$$|\gamma(s)-\eta(s)|\le|\gamma(s)-\gamma(0)|+s\left|\frac{\gamma(\varepsilon)-\gamma(0)}{\varepsilon}\right|\le M\varepsilon+Ms\le 2M\varepsilon,$$

for every $s \in [0, \varepsilon]$, and similarly (by replacing γ with γ' and using that $x'(s)$ and $y'(s)$ belong to C^1) we can estimate

$$|\gamma'(s) - \eta'(s)| \leq \left| \gamma'(s) - \frac{\gamma(\varepsilon) - \gamma(0)}{\varepsilon} \right| \leq M\varepsilon.$$

Consider now any C^2-curve $\gamma : [0,1] \to \mathbb{C}$. For $n \geq 2$ let η be the piecewise affine curve that linearly interpolates between $\gamma(k/n)$ with $\gamma((k+1)/n)$, $k = 0, \dots, n-1$. Setting again $M := \max\{|\gamma'(s)| + |\gamma''(s)| : s \in [0,1]\}$ we get by repeating the previous estimates over intervals of length $\varepsilon = 1/n$ that

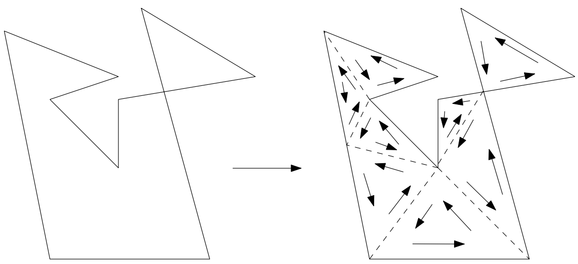
$$\sup_{s \in [0,1]} |\gamma(s) - \eta(s)| = \max_{k=0,\dots,n-1} \sup_{s \in [k/n,(k+1)/n]} |\gamma(s) - \eta(s)| \leq \frac{2M}{n},$$

and similarly $\sup_{s \in [0,1]} |\gamma'(s) - \eta'(s)| \leq M/n$. Letting $n \to +\infty$ we obtain the absolute convergence of η to γ.

(b) Show taking advantage of Goursat Theorem that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, f' is continuous, and γ is like in the statement of the exercise, then

$$\int_\gamma f \, dz = 0.$$

SOL: First of all notice that $\int_{\gamma_n} f \, dz = 0$ for every $n \in \mathbb{N}$. This follows by Goursat theorem, since integration over a polynomial closed curve can be decomposed as the finite sum of integration over triangles.



Hence, we are left to prove

$$\left| \int_\gamma f \, dz \right| \leq \left| \int_{\gamma_n} f \, dz \right| + \left| \int_{\gamma_n} f \, dz - \int_\gamma f \, dz \right| = \left| \int_{\gamma_n} f \, dz - \int_\gamma f \, dz \right| \to 0,$$

as $n \to +\infty$. Without loss of generality we can suppose $|\gamma'| = 1$ by choosing arc-length parametrization and $|f(z)| + |f'(z)| \leq 1$ uniformly on a neighbourhood of γ, just by multiplying f by a constant. Then

$$\begin{aligned}
&\left|\int_{\gamma_n} f\,dz - \int_\gamma f\,dz\right| = \left|\int_a^b f(\gamma_n(s))\gamma_n'(s) - f(\gamma(s))\gamma'(s)\,dz\right| \\
&\leq \int_a^b |f(\gamma_n(s))||\gamma'(s) - \gamma_n'(s)|\,ds + \int_a^b |f(\gamma(s)) - f(\gamma_n(s))||\gamma'(s)|\,ds \\
&\leq (b-a)\max_{s\in[a,b]}|\gamma'(s) - \gamma_n'(s)| + (b-a)\max_{s\in[a,b]}|\gamma(s) - \gamma_n(s)|,
\end{aligned}$$

where in the last line we took advantage again of the Mean Value Theorem on f: $|f(\gamma(s)) - f(\gamma_n(s))| \leq |\gamma(s) - \gamma_n(s)| \cdot \max\{f'(z) : z \text{ in a neighbourhood of } \gamma\}$. By assumption, we point out that the above integral converges to zero as $n \to +\infty$.