Exercises with a $\star$ are eligible for bonus points.

4.1. Cauchy Formula Compute the following integrals

(a) $\int_0^{2\pi} e^{e^{it}}\,dt$.

SOL: Express this as a line integral over the unit circle $z = e^{it}$:

$$\int_0^{2\pi} e^{e^{it}}\,dt = \int_0^{2\pi} \frac{e^{e^{it}}}{e^{it}} e^{it}\,dt = -i\int_0^{2\pi} \frac{e^{e^{it}}}{e^{it}} ie^{it}\,dt = -i\int_{|z|=1} \frac{e^z}{z}\,dz.$$

By the Cauchy integral formula, we get that it is equal to $-i(2\pi i)e^0 = 2\pi$.

(b) $\int_\gamma \frac{iz}{(z-\alpha)(z+i\alpha)}\,dz$, when $\gamma = \{z \in \mathbb{C} : |z| = 2\}$ and $\alpha \in [-1, 1]$.

SOL: We have

$$\int_\gamma \frac{iz}{(z-\alpha)(z+i\alpha)}\,dz = \frac{1}{2}\int_\gamma \frac{i-1}{z+i\alpha} + \frac{1+i}{z-\alpha}\,dz = \frac{2\pi i}{2}(i-1+1+i) = -2\pi,$$

by applying the Cauchy integral formula to both terms in the last integral.

(c) $\int_\gamma \frac{z^2+z}{z^2+1}\,dz$, when $\gamma = \{z \in \mathbb{C} : |z| = 3\}$.

SOL: We have that

$$\begin{aligned}\int_\gamma \frac{z^2+z}{z^2+1}\,dz &= \int_\gamma \frac{z(z+1)}{(z+i)(z-i)}\,dz = \frac{1}{2}\int_\gamma \frac{z(1+i)}{z+i} + \frac{z(1-i)}{z-i}\,dz \\ &= \frac{2\pi i}{2}(-i(1+i) + i(1-i)) = \pi i(-i+1+i+1) = 2\pi i,\end{aligned}$$

by applying the Cauchy integral formula to both terms in the last integral.

(d) $\int_\gamma \sin(z)^2\cos(z)\,dz$, when γ is the 'infinity symbol' $t \mapsto \sin(t) + i\sin(t)\cos(t)$, $t \in [0, 2\pi]$.

SOL: Notice that $f(z) = \sin(z)^2\cos(z)$ is holomorphic, and γ is closed. Hence, by Cauchy Theorem, the above integral is equal to zero. Alternatively, notice that $F(z) = \sin^3(z)/3$ is a primitive of f, and therefore the integral depends only on the end points by the Fundamental Theorem of Calculus, and since γ is closed, the integral must vanish.

4.2. The complex logarithm Let

$$U = \mathbb{C} \setminus \{z \in \mathbb{C} : \Im(z) = 0, \Re(z) \le 0\}$$

be the open set obtained by removing the negative real axis from the complex plane $\mathbb{C}$. The complex logarithm is defined in U as

$$\log(z) := \log(|z|) + i\arg(z), \quad z = |z|e^{i\arg(z)},$$

where $\arg(z) \in]-\pi, \pi[$. Show that for every $z \in U$

$$\log(z) = \int_\gamma \frac{1}{w}\,dw,$$

where γ is the segment connecting 1 to z.

Hint: integrate over a well chosen closed curve containing γ and passing through $|z|$.

SOL: Fix $z \in U$ and let $\theta = \arg(z) \in]-\pi, \pi[$ so that $z = |z|e^{i\theta}$. Then, consider the closed curve σ defined as the concatenation of: $\gamma_1(t) = (1-t) + t|z|$, $t \in [0,1]$ (the segment joining 1 with $|z|$), then $\gamma_2(t) = |z|e^{it}$ for $t \in [0, \theta]$ (the arc centred at the origin connecting $|z|$ to z), and finally γ^-, (the segment joining z to 1). Since $w \mapsto 1/w$ is holomorphic in U, by Cauchy Theorem we have by integrating over σ that

$$\int_\gamma \frac{1}{w}\,dw = \int_{\gamma_1} \frac{1}{w}\,dw + \int_{\gamma_2} \frac{1}{w}\,dw = \int_0^1 \frac{|z|-1}{(1+t)+t|z|}\,dt + \int_0^\theta \frac{i|z|e^{it}}{|z|e^{it}}\,dt$$
$$= \log((1+t)+t|z|)\big|_{t=0}^{t=1} + i\theta = \log(|z|) + i\arg(z)$$

which is exactly the definition of the complex logarithm.

4.3. Quotients and integration Let Ω be an open subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be an holomorphic function, and $\gamma : [a,b] \to \Omega$ a smooth, closed curve. Suppose $|f(z) - 1| < 1$ for all $z \in \Omega$. Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0.$$

SOL: Since $f(z)$ lies in the *open* ball of radius 1 centered in 1, we deduce that $f(z) \neq 0$ for all $z \in \Omega$. This implies that f'/f is holomorphic and well defined in Ω. Hence, by Cauchy Theorem, the integral along the closed curve γ must be equal to zero.

4.4. Generalized Liouville Prove the following theorem: every holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that

$$|f(w)| \le c|w|^n, \quad \text{for all } w \in \{z \in \mathbb{C} : |z| > C\}$$

for some $c, C > 0$ and $n \geq 0$, is a complex polynomial of degree at most n.

SOL: Let f, $n \geq 0$, and $C, c > 0$ like in the above statement. Fix $z_0 \in \mathbb{C}$, let $R > |z_0| + C$ and set $k > n$. By Cauchy integral formula we have that

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^{k+1}}\, dz,$$

implying that

$$\begin{aligned}
|f^{(k)}(z_0)| &= \left| \frac{k!}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^{k+1}}\, dz \right| \leq \frac{k!}{2\pi} \int_{|z-z_0|=R} \frac{|f(z)|}{|z-z_0|^{k+1}}\, dz \\
&\leq \frac{k!}{2\pi} \int_{|z-z_0|=R} \frac{c|z|^n}{|z-z_0|^{k+1}}\, dz \\
&= \frac{k!}{2\pi} \int_0^{2\pi} \frac{c|Re^{it} + z_0|^n}{R^k}\, dt \\
&< \frac{k!}{2\pi} \int_0^{2\pi} \frac{2^n c R^n}{R^k}\, dt = \frac{c 2^n (n+1)!}{R^{k-n}},
\end{aligned}$$

where in the last inequality we used that $R > |z_0|$ to estimate $|Re^{it} + z_0|^n \leq (|Re^{it}| + |z_0|)^n < (2R)^n = 2^n R^n$. Since R is arbitrary and $k - n \geq 1$, letting $R \to +\infty$ we get that $|f^k(z_0)| = 0$. Hence, $f^{(k)} \equiv 0$ in $\mathbb{C}$, since z_0 is also arbitrary. Therefore, expressing f as a power serie $f(z) = \sum_{k=0}^{+\infty} a_k z^k$ we get that $a_k = 0$ for all $k > n$, implying that f must be a complex polynomial of degree at most n.