Exercises with a $\star$ are eligible for bonus points.

5.1. Discrete maps A subset $\mathcal{A}$ of an domain $\Omega \subset \mathbb{C}$ is called *discrete* in Ω if it has no limit point in Ω. A function $f : \Omega \to \mathbb{C}$ is called *discrete* if for every $w \in \mathbb{C}$ the set

$$E_w := \{z \in \Omega : f(z) = w\}$$

is discrete in Ω.

(a) Let Ω be connected and open. Show that every non-constant holomorphic function $f : \Omega \to \mathbb{C}$ is discrete.

SOL: Fix $w \in \Omega$ and define $g : \Omega \to \mathbb{C}$ as $g(z) = f(z) - w$. Then $E_w = \{z \in \Omega : f(z) = w\} = \{z \in \Omega : f(z) - w = 0\} = \{z \in \Omega : g(z) = 0\}$. Since g is also non-constant and holomorphic, we know that its set of zeros has to be discrete (all zeros are isolated), and consequently E_w is also discrete.

(b) Show that if Ω is compact, then $\mathcal{A} \subset \Omega$ is discrete if and only if it has finite cardinality. Is this true if Ω is merely bounded?

SOL: If the cardinality of $\mathcal{A}$ if finite, then all points are isolated and hence the set is discrete. Suppose now $\mathcal{A}$ is discrete and Ω is compact. Suppose on the contrary $\mathcal{A}$ is infinite and that there exists an injection $j : \mathbb{N} \to \mathcal{A}$. The associated sequence $x_n = j(n)$ has the property to take all different values and to be bounded since it is in particular a sequence in Ω which is bounded. By Bolzano-Weierstrass, there exist a subsequence x_{n_k} converging to x_∞ in $\bar{\Omega}$. If Ω is compact, by Heine-Borel $\bar{\Omega} = \Omega$ and hence x_∞ is a limit point. But this contradicts the assumption that $\mathcal{A}$ is discrete Hence there is no injection j and $\mathcal{A}$ has finite cardinality. On the other hand, if Ω is only bounded, the limit point x_∞ might belong to $\mathbb{C} \setminus \Omega$, and hence it does not contradict the existence of j. For instance the set $\{1/n : n \in \mathbb{N}\}$ is not discrete in $[0, 1]$ but it is discrete in $(0, 1]$.

5.2. Order of zeros

(a) Find the zeros of the function $z \mapsto \cos(z^2)$ and determine their order.

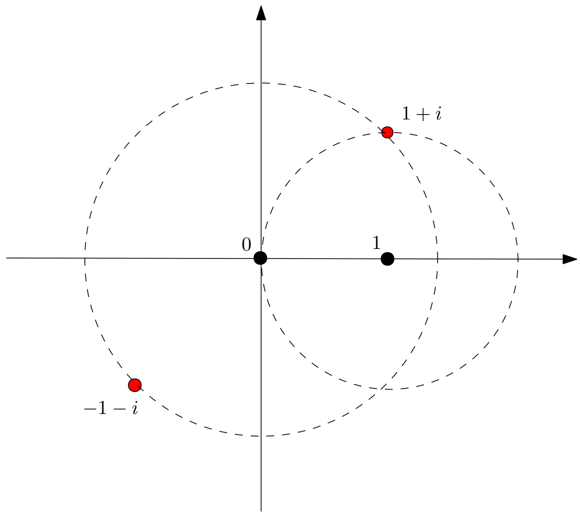
SOL: Taking advantage of the definition of complex cosine, we have that $\cos(z_0^2) = 0$ if $z_0 = \pm\sqrt{\frac{\pi(1+2k)}{2}}$ or $z_0 = \pm i\sqrt{\frac{\pi(1+2k)}{2}}$ for $k \geq 0$. Since $(\cos(z^2))' = -2z\sin(z^2)$ is different from zero when evaluated in z_0 we deduce that $\mathrm{ord}_{z_0}(\cos(z^2)) = 1$.

(b) Let $f, g : \mathbb{C} \to \mathbb{C}$ two holomorphic functions that vanish simultaneously at some point $z_0 \in \mathbb{C}$ with order $a \in \mathbb{N}$ and $b \in \mathbb{N}$ respectively. Show that the function $h = f + g$ vanish at z_0 with order $c \geq \min\{a, b\}$. Give an explicit example realizing the strict inequality.

SOL: By definition of order $f(z_0) = \cdots = f^{(a-1)}(z_0) = g(z_0) = \cdots = g^{(b-1)}(z_0) = 0$, $f^{(a)}(z_0) \neq 0$ and $g^{(b)}(z_0) \neq 0$. It follows directly that $h^{(k-1)}(z_0) = 0$ for $k = 0, \ldots, \min\{a, b\} - 1$ by linearity of the differentiation, showing that $c \geq \min\{a, b\}$. Notice that if $g = -f$ then h has order infinity at z_0 because $h \equiv 0$.

5.3. Taylor series Compute the radius of convergence of the Taylor serie of the function $f(z) = \frac{\sin(z)}{z^2 - i}$ in $z_0 = 0$ and $z_0 = 1$.

SOL: We note the two singularities: $z^2 + i = 0$, when $z = i + 1$ or $z = -1 - i$. Also, $\sin(1 + i) \neq 0$ and $\sin(-1 - i) \neq 0$. Therefore, the radius of convergence of the Taylor series[1] in $z_0 = 0$ is $|1 + i| = |-1 - i| = \sqrt{2}$. In $z_0 = 1$ the radius of convergence is $\min\{|(1 + i) - 1|, |(-1 - i) - 1|\} = 1$.



5.4. A complex ODE Take advantage of the power series expansion to find $f : \mathbb{C} \to \mathbb{C}$ holomorphic such that $f'(z) = z^2 f(z)$ and $f(0) = 1$.

SOL: Write $f(z) = \sum_{n \geq 0} a_n z^n$. Then,

$$f'(z) = z^2 f(z) \Leftrightarrow \sum_{n \geq 1} n a_n z^{n-1} = \sum_{n \geq 0} a_n z^{n+2}.$$

[1] Recall that if $f : \Omega \to \mathbb{C}$ is holomorphic and $\{z \in \mathbb{C} : |z - z_0| < r\} \subset \Omega$, then the Taylor serie of f in z_0 has radius of convergence at least r.

Shifting the indices we get that

$$a_1 + 2a_2 z + \sum_{n\geq 1}(n+1)a_{n+1}z^n = \sum_{n\geq 2} a_{n-2}z^n.$$

Since $f(0) = 1$, this impose the following relations on the coefficients

$$\begin{cases} a_0 = 1,\ a_1 = a_2 = 0, & \\ a_{n+1} = \frac{a_{n-2}}{n+1}, & n \geq 2. \end{cases}$$

From $a_{n+1} = \frac{a_{n-2}}{n+1}$ when $n \geq 2$ we get $a_n = \frac{a_{n-3}}{n}$ for all $n \geq 3$. In particular

$$a_3 = \frac{a_0}{3} = \frac{1}{3}.$$

Hence, if $n = 3k$ we get that

$$a_{3k} = \frac{a_{3(k-1)}}{3k} = \frac{a_{3(k-2)}}{3k(3(k-1))} = \cdots = \frac{a_3}{3^{k-1}k!} = \frac{1}{3^k k!}.$$

On the other side, if n is not a multiple of k one can easily prove by induction that $a_n = 0$ taking as base case $a_1 = a_2 = 0$. Finally we get

$$f(z) = \sum_{k\geq 0} \frac{z^{3k}}{3^k k!} = \sum_{k\geq 0} \frac{(z^3/3)^k}{k!} = e^{z^3/3}.$$

5.5. Riemann continuation Theorem Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be holomorphic. Show that the following are equivalent:

1. There exists $g : \mathbb{C} \to \mathbb{C}$ holomorphic, such that $g(z) = f(z)$ for all $z \neq 0$.
2. There exists $g : \mathbb{C} \to \mathbb{C}$ continuous, such that $g(z) = f(z)$ for all $z \neq 0$.
3. There exists $\varepsilon > 0$ such that f is bounded in $\dot{B}_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\} \setminus \{0\}$.
4. $\lim_{z\to 0} zf(z) = 0$.

Hint: to prove 4. $\Rightarrow$ 1. *define* $h(z) = zf(z)$ *when* $z \neq 0$ *and* $h(0) = 0$. *Analyse the relation between* $f(z)$, $h(z)$ *and* $k(z) := zh(z)$.

SOL: Notice that the implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. are elementary. We are left to show 4. $\Rightarrow$ 1. Introduce the function

$$h(z) := \begin{cases} zf(z), & z \neq 0, \\ 0, & z = 0, \end{cases}$$

and set $k(z) = zh(z)$. By assumption 4. h and k are holomorphic in $\mathbb{C} \setminus \{0\}$ and continuous in the whole complex plane $\mathbb{C}$. Since $k(z) = k(0) + zh(z)$ we deduce that k is complex differentiable in zero and hence holomorphic in $\mathbb{C}$. By Taylor representation of holomorphic functions, $k(z) = a_0 + a_1 z + a_2 z^2 + \dots$ for coefficients $a_0, a_1, \dots \in \mathbb{C}$. Since $k(0) = 0$ and $k'(0) = h(0) = 0$ we deduce that $k(z) = a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots = z^2(a_2 + a_3 z + a_4 z^2 + \dots)$. Now, recalling that $k(z) = z^2 f(z)$ for $z \neq 0$ we deduce that $g(z) := a_2 + a_3 z + a_4 z^2 + \dots$ is indeed an holomorphic extension of f in $\mathbb{C}$.