Exercises with a $\star$ are eligible for bonus points.

6.1. Uniform convergence Let Ω be an open subset of $\mathbb{R}^M$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C^0(\Omega, \mathbb{R}^N)$. Show that if f_n converges locally uniformly[1] to some map $f : \Omega \to \mathbb{R}^N$, then $f \in C^0(\Omega, \mathbb{R}^N)$.

SOL: It suffices to show that f is continuous in an arbitrarily point $x \in \Omega$. Let $\varepsilon > 0$. Then, by local uniform convergence there exists an open neighbourhood U of x in Ω and $n_0 \in \mathbb{N}$ such that $|f_n(y) - f(y)| < \varepsilon/3$ for all $y \in U$ and $n \geq n_0$. I particular, since f_{n_0} is continuous in x, there exists $\delta > 0$ small enough such that $|x - y| < \delta$ implies $|f_{n_0}(x) - f_{n_0}(y)| < \varepsilon/3$ and $y \in U$. Hence, we showed that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in \Omega$ satisfying $|x - y| < \delta$

$$|f(x) - f(y)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f(y)| < \varepsilon,$$

proving continuity of f in $x \in \Omega$, and consequently in all Ω.

6.2. A generalization In Exercise 3.2 we showed the following identity holding for all polynomial p an circle $\gamma = \{z \in \mathbb{C} : |z - z_0| = r\}$: $\int_\gamma p(\bar{z})\, dz = 2\pi i r^2 p'(\bar{z}_0)$. Extend this result to all entire functions $f : \mathbb{C} \to \mathbb{C}$, proving that

$$\int_\gamma f(\bar{z})\, dz = 2\pi i r^2 f'(\bar{z}_0)$$

for all circle γ of radius $r > 0$ centered in z_0.

SOL: We have an absolutely convergent series expansion $f(z) = \sum_{n\in\mathbb{N}} a_n (z - \overline{z_0})^n$ for z in the ball centered in $\bar{z}_0$ of radius $r + \varepsilon$ for some $\varepsilon > 0$. Define

$$g(z) = \sum_{n\in\mathbb{N}} \overline{a_n} (z - z_0)^n, \text{ for } |z - z_0| < r + \varepsilon.$$

[1] Recall that local uniform convergence means that for every $x \in \Omega$ there exists an open neighbourhood U of x in Ω so that $f_n \to f$ uniformly in U: $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$, such that for all $n \geq N \Rightarrow |f_n(y) - f(y)| < \varepsilon$ for all $y \in U$.

Then g is analytic where it is defined. We also have $f(\overline{z}) = \overline{g(z)}$ hence

$$\begin{aligned}
\int_{|z-z_0|=r} f(\overline{z})dz &= \int_{|z-z_0|=r} \overline{g(z)}dz \\
&= \overline{\int_{|z-z_0|=r} g(z)\overline{dz}} \\
&= \overline{\int_0^{2\pi} g(z_0+re^{it})\overline{ire^{it}}dt} \\
&= -r^2\overline{\int_0^{2\pi} \frac{g(z_0+re^{it})}{(re^{it})^2} ire^{it}dt} \\
&= -r^2\overline{\int_{|z-z_0|=r} \frac{g(z)}{(z-z_0)^2}dz}.
\end{aligned}$$

By Cauchy's formula (the general one involving derivatives), the last integral equals

$$2\pi i g'(z_0) = 2\pi i\overline{a_1} = 2\pi i\overline{f'(\overline{z_0})},$$

hence we get

$$\int_{|z-z_0|=r} f(\overline{z})dz = -r^2\overline{2\pi i\overline{f'(\overline{z_0})}} = 2\pi i r^2 f'(\overline{z_0}).$$

Alternatively, one can take advantage of Exercise 3.2 by carefully commuting sum and integration taking advantage of Fubini's Theorem (see solutions of Exercise 2.4 for the statement).

6.3. Schwarz reflection principle Let Ω be open, connected,‘ and symmetric with respect to the x-axis (i.e. $z \mapsto \bar{z}$ preserves Ω), and let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose that $L := \{z \in \Omega : \Im(z) = 0\}$ is non-empty. Prove that $f(\bar{z}) = \overline{f(z)}$ for all $z \in \Omega$ if and only if f is real valued on L.

Hint: consider g to be the restriction of f to the upper half plane intersected with Ω. 'Reflect' g by imposing $g^*(z) := \overline{g(\bar{z})}$. Argue taking advantage of Morera's Theorem.

SOL: One direction is elementary: since $z \in L$ implies $\bar{z} = z$, the relation $\overline{f(z)} = f(\bar{z})$ gives on L that $\overline{f(z)} = f(z)$, and hence $2\Im(f(z)) = 0$, showing that f has real image on L. To prove the other direction suppose f real valued on L. Define the function

$$h(z) := \begin{cases} f(z), & \text{if } z \in \Omega, \Im(z) \geq 0 \\ \overline{f(\bar{z})}, & \text{if } z \in \Omega, \Im(z) < 0. \end{cases}$$

We claim that h is continuous. In fact, by construction of f we only have to check continuity approaching L, that is

$$\lim_{\substack{\Im(z)\to 0^+ \\ z\in\Omega}} h(z) = \lim_{\substack{\Im(z)\to 0^- \\ z\in\Omega}} h(z) \Leftrightarrow \lim_{\substack{\Im(z)\to 0^+ \\ z\in\Omega}} f(z) = \lim_{\substack{\Im(z)\to 0^- \\ z\in\Omega}} \overline{f(\bar{z})}.$$

This holds because by assumption $\overline{f(\bar{z})} = f(z)$ for $z \in L$ and the conjugation $w \mapsto \bar{w}$ is continuous. We prove now that h is holomorphic: by Morera's Theorem we have to check that $\int_T h\,dz = 0$ for every triangle $T \subset \Omega$. We split this into three cases:

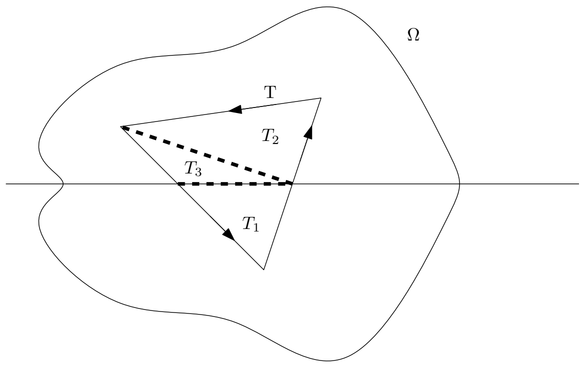
- Type 1: $T \subset (\Omega \cap \{z : \Im(z) \geq 0\})$. In this case $h = f$ on T and it suffices to apply Goursat's Theorem for holomorphic functions.
- Type 2: $T \subset (\Omega \cap \{z : \Im(z) < 0\})$. In this case we can compute

$$\int_T h(z)\,dz = \int_T \overline{f(\bar{z})}\,dz = \overline{\int_T f(\bar{z})\,\overline{dz}} = \overline{\int_{\bar{T}} f(z)\,dz} = \bar{0} = 0,$$

  since $\bar{T}$ is a triangle in the upper halfplane, and therefore we can apply the result for Type 1.
- Generic type: T is a generic triangle in Ω. In this case one can check that there exist (at most) 3 oriented triangles T_1, T_2, and T_3, all of them of type 1 or 2 such that

$$\int_T h\,dz = \int_{T_1} h\,dz + \int_{T_2} h\,dz + \int_{T_3} h\,dz.$$

  Hence, by applying the previous cases, we deduce finally $\int_T h\,dz = 0$.



To conclude, observe that we constructed h holomorphic in Ω that agrees with f when $\Im(z) \geq 0$. By the uniqueness of analytic extensions, we deduce that $f = h$, and hence $f(\bar{z}) = h(\bar{z}) = \overline{h(z)} = \overline{f(z)}$ as wished.

6.4. Dense image Show that the image of an non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is *dense* in $\mathbb{C}$, that is: for every $z \in \mathbb{C}$ and $\varepsilon > 0$, there exists $w \in \mathbb{C}$ such that $|z - f(w)| < \varepsilon$.

Remark: The little Picard Theorem asserts in fact that $f(\mathbb{C})$ misses at most one single point of $\mathbb{C}$!

SOL: By contradiction suppose that there exists $z^* \in \mathbb{C}$ and $\varepsilon^* > 0$ such that $f(w) \notin \{z \in \mathbb{C} : |z - z^*| < \varepsilon^*\}$ for all $w \in \mathbb{C}$. Define the function

$$g(w) := \frac{1}{f(w) - z^*}.$$

By assumption $|f(w) - z^*| \geq \varepsilon^*$ and hence g is a well defined holomorphic function since the denominator is never zero. On the other side

$$|g(w)| = \frac{1}{|f(w) - z^*|} \leq \frac{1}{\varepsilon^*}, \quad \forall w \in \mathbb{C},$$

contradicting Liouville's Theorem (every holomorphic function on $\mathbb{C}$ is either constant or unbounded). Hence, for all $\varepsilon > 0$ and $z \in \mathbb{C}$ there exists $w \in \mathbb{C}$ such that $|f(w) - z| < \varepsilon$, proving the density of the image of f in $\mathbb{C}$.

6.5. Geometric identity Let $\Omega \subset \mathbb{C}$ be bounded and open with C^1-boundary. Show that

$$\int_{\partial\Omega} \bar{z}\, dz = 2iA(\Omega),$$

where $A(\Omega)$ denotes the measure of the area of the set Ω.

SOL: Parametrize $\partial\Omega$ with a C^1-curve $\gamma(s) = x(s) + iy(s)$, $s \in [0, \ell]$ for some $\ell > 0$. Then, one can compute

$$\begin{aligned}
\int_{\partial\Omega} \bar{z}\, dz &= \int_0^\ell (x(s) - iy(s))(x'(s) + iy'(s))\, ds \\
&= \int_0^\ell \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \cdot \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} + i \begin{pmatrix} -y(s) \\ x(s) \end{pmatrix} \cdot \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} ds \\
&= \int_0^\ell \frac{1}{2} \frac{d}{ds} |\gamma(s)|^2 + i \begin{pmatrix} -y(s) \\ x(s) \end{pmatrix} \cdot \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} ds \\
&= \frac{1}{2}(|\gamma(\ell)|^2 - |\gamma(0)|^2) + i \int_{\partial\Omega} (-y, x) \cdot dr.
\end{aligned}$$

The first term is equal to zero because γ is closed and hence $\gamma(\ell) = \gamma(0)$. The second term can be treated with Green's Theorem (see footnote Exercise sheet 2) obtaining

$$\int_{\partial\Omega} \bar{z}\, dz = i \int_{\partial\Omega} (-y, x) \cdot dr = i \iint_\Omega \partial_x x + \partial_y y\, dx\, dy = i \iint_\Omega 2\, dx\, dy = 2iA(\Omega).$$