Exercises with a $\star$ are eligible for bonus points.

7.1. Calculus of residues Determine the order of the poles of the following functions and compute their residue at the indicated points:

$$\operatorname{res}_{2i}\left(\frac{1}{z^2+4}\right), \quad \operatorname{res}_0\left(\frac{\sin(z)}{z^2}\right), \quad \operatorname{res}_0\left(\frac{\cos(z)}{z^2}\right), \quad \operatorname{res}_1\left(\frac{1}{z^5-1}\right).$$

SOL: Since $(1/(z^2+4))^{-1} = (z^2+4) = (z-2i)(z+2i)$ has a zero of order 1 in $2i$, we get that $1/(z^2+4)$ has a pole of order 1 at the same point. The residue is given by

$$\operatorname{res}_{2i}\frac{1}{z^2+4} = \lim_{z\to 2i}(z-2i)\frac{1}{z^2+4} = \frac{1}{4i}.$$

For the second function, taking advantage of the Taylor expansion of $\sin(z)$ at 0 we have that

$$\frac{\sin(z)}{z^2} = z^{-2}\sum_{k=0}^{+\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!} = z^{-1} + \sum_{k=1}^{+\infty}\frac{(-1)^k z^{2k-1}}{(2k+1)!},$$

showing at the same time that the pole at zero is of order 1 and the residue is

$$\operatorname{res}_0\frac{\sin(z)}{z^2} = 1.$$

We argue similarly for the third function:

$$\frac{\cos(z)}{z^2} = z^{-2}\sum_{k=0}^{+\infty}\frac{(-1)^k z^{2k}}{(2k)!} = z^{-2} + \sum_{k=1}^{+\infty}\frac{(-1)^k z^{2k-2}}{(2k)!},$$

and hence the pole is of order 2, and

$$\operatorname{res}_0\frac{\cos(z)}{z^2} = 0.$$

Finally, $(1/(z^5-1))^{-1} = z^5-1$ has a zero of order 1 in 1, and therefore the pole of $1/(z^5-1)$ is also of order 1 by definition. The residue is

$$\operatorname{res}_1\frac{1}{z^5-1} = \lim_{z\to 1}\frac{(z-1)}{z^5-1} = \lim_{z\to 1}\frac{1}{5z^4} = \frac{1}{5},$$

where we took advantage of Bernoulli-l'Hôpital's rule to compute the limit.

7.2. Complex integrals Compute the following complex integrals taking advantage of the Residue Theorem.

(a)

$$\int_{|z|=2} \frac{e^z}{z^2(z-1)}\, dz.$$

SOL: The poles of $f(z) = e^z/(z^2(z-1))$ are at 0 and 1, with multiplicity 2 and 1 respectively. We compute

$$\operatorname{res}_0(f) = \lim_{z\to 0}(z^2 f(z))' = \lim_{z\to 0}\Big(\frac{e^z}{(z-1)}\Big)' = -2,$$

and

$$\operatorname{res}_1(f) = \lim_{z\to 1}(z-1)f(z) = e.$$

By the Residue Theorem, since 0 and 1 are in the interior of the disc of radius 2 centered at the origin, we can compute

$$\int_{|z|=2} \frac{e^z}{z^2(z-1)}\, dz = 2\pi i(e-2).$$

(b)

$$\int_{|z|=1} \frac{1}{z^2(z^2-4)e^z}\, dz.$$

SOL: The poles of $f(z) = 1/(z^2(z^2-4))$ are at 0, $\sqrt{2}$ and $-\sqrt{2}$ with multiplicity 2, 1, 1 respectively. However, since only 0 belongs to the interior of the circumference of radius 1 centered at the origin, we get that

$$\int_{|z|=1} \frac{1}{z^2(z^2-4)e^z}\, dz = 2\pi i \operatorname{res}_0(f) = 2\pi i \lim_{z\to 0}(z^2 f(z))' = 2\pi i \lim_{z\to 0}\Big(\frac{1}{(z^2-4)e^z}\Big)' = \frac{\pi i}{2}.$$

(c)

$$\int_{|z|=1/2} \frac{1}{z\sin(1/z)}\, dz.$$

SOL: By parametrizing the contour as $t \mapsto e^{it}/2$ we get that

$$\int_{|z|=1/2} \frac{1}{z\sin(1/z)}\, dz = \int_0^{2\pi} \frac{ie^{it}/2}{e^{it}/2\sin(2e^{-it})}\, dz = \int_{|w|=2} \frac{1}{w\sin(w)}\, dw$$

where we recognised $t \mapsto 2e^{-it}$ as the circle of radius 2 oriented in the *clockwise* direction (hence the change of sign). The only pole contained in $|w| \leq 2$ is $z = 0$, and it is of order 2 (arguing like in Exercise 7.1). We get

$$\int_{|w|=2} \frac{1}{w\sin(w)}\, dw = 2\pi i \operatorname{res}_0\Big(\frac{1}{w\sin(w)}\Big) = 2\pi i \lim_{w\to 0}\Big(\frac{w^2}{w\sin(w)}\Big)' = 0.$$

(d)

$$\int_\gamma \frac{1}{(z-i)(z+2)(z-4)}\, dz,$$

for any simple closed curve γ that does not intersect the points $\{i, -2, 4\}$.

SOL: The poles of $f(z) = \frac{1}{(z-i)(z+2)(z-4)}$ are all of order 1, and equal to i, -2, and 4. The associated residues are

$$\begin{aligned} \mathrm{res}_i\, f &= \frac{1}{(2+i)(i-4)}, \\ \mathrm{res}_{-2}\, f &= \frac{1}{6(2+i)}, \\ \mathrm{res}_4\, f &= \frac{1}{6(4-i)}. \end{aligned}$$

The value of the integral depends on weather or not the poles are inside the curve γ:

$$\int_\gamma \frac{1}{(z-i)(z+2)(z-4)}\, dz = 2\pi i\Big(\frac{\mathbf{1}_\gamma(i)}{(2+i)(i-4)} + \frac{\mathbf{1}_\gamma(-2)}{6(2+i)} + \frac{\mathbf{1}_\gamma(4)}{6(4-i)}\Big),$$

where here $\mathbf{1}_\gamma : \mathbb{C} \to \{0,1\}$ is the indicator function defined as

$$\mathbf{1}_\gamma(z) = \begin{cases} 1, & \text{if } z \text{ belongs to the open interior of } \gamma, \\ 0, & \text{otherwise.} \end{cases}$$

7.3. Poles at infinity Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. We say that f *has a pole at infinity of order* $N \in \mathbb{N}$ if the function $g(z) := f(1/z)$ has a pole of order N at the origin in the usual sense. Prove that if $f : \mathbb{C} \to \mathbb{C}$ has a pole of order $N \in \mathbb{N}$ at infinity, then it has to be a polynomial of degree $N \in \mathbb{N}$.

SOL: Since f is holomorphic, the expansion

$$f(z) = \sum_{k=0}^{+\infty} a_k z^k,$$

converges in any ball centered in 0. If f has a pole at infinity of order N, by definition for $z \neq 0$

$$g(z) = f(1/z) = \sum_{k=0}^{+\infty} a_k z^{-k}$$

has a pole of order N at zero, which means

$$z^N g(z) = \sum_{k=0}^{+\infty} a_k z^{N-k}$$

is holomorphic in a neighbourhood of 0. This implies that $a_k = 0$ for every $k > N$ and $a_N \neq 0$, proving that

$$f(z) = \sum_{k=0}^{N} a_k z^k,$$

that is, f is a polynomial of degree N as claimed.

7.4. The Gamma function Let $Z_- := \{0, -1, -2, \dots\}$ the set of all non-positive integers, and define for all $\tau \in \mathbb{R}$ the set $U_\tau := \{z \in \mathbb{C} : \Re(z) > \tau, z \notin Z_-\}$, and $U := \mathbb{C} \setminus Z_-$.

(a) Show that the function defined by the complex improper Riemann integral

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1}\, dt$$

is well defined for all $z \in U_1$. (Here $t^{z-1} = \exp((z-1)\log(t))$).

SOL: First of all, fix $z \in U_1$. Then, $\Re(z-1) > 0$ by definition of U_1, and therefore there exists $a > 0$ large enough (depending on $\Re(z)$) such that $\Re(z-1)\log(t) < t/2$ for all $t > a$ (this follows from the elementary observation $\lim_{s\to+\infty} \log(s)/s = 0$). Now for every $n > a$ one has that

$$\begin{aligned}
\left|\int_a^n e^{-t} t^{z-1}\, dt\right| &= \left|\int_a^n e^{-t} e^{(\Re(z-1)+i\Im(z-1))\log(t)}\, dt\right| \\
&= \left|\int_a^n e^{-t} e^{i\Im(z-1)\log(t)} e^{\Re(z-1)\log(t)}\, dt\right| \\
&\leq \int_a^n e^{-t} |e^{i\Im(z-1)\log(t)}||e^{\Re(z-1)\log(t)}|\, dt = \int_a^n e^{-t} e^{\Re(z-1)\log(t)}\, dt \\
&\leq \int_a^n e^{-t} e^{t/2}\, dt = [-2e^{-t/2}]_{t=a}^{t=n} = -2e^{-a/2} + 2e^{-n/2}.
\end{aligned}$$

On the other side, notice that on the interval $[0, a]$ the function $t \mapsto e^{-t+\Re(z-1)\ln(t)}$ is continuous, and therefore the integral $\int_0^a |e^{-t} t^{z-1}|\, dt =: \alpha$ is well defined. Hence, we conclude that the improper integral defining Γ converges absolutely:

$$\lim_{n\to+\infty} \int_0^n |e^{-t} t^{z-1}|\, dt = \int_0^a |e^{-t} t^{z-1}|\, dt + \lim_{n\to+\infty} \int_a^n |e^{-t} t^{z-1}|\, dt \leq \alpha - 2e^{-a/2} < +\infty$$

proving that $\Gamma(z)$ is well defined for all $z \in U_1$.

(b) Prove that Γ is holomorphic in U_1.

Hint: First show that the functions of the sequence $(\Gamma_n)_{n\in\mathbb{N}}$ given by truncating the integral at height n ($\Gamma_n(z) = \int_0^n e^{-t}t^{z-1}\,dt$) are holomorphic. Then, show that $\Gamma_n \to \Gamma$ uniformly in all compact subsets of U_1.

SOL: Define the sequence

$$(\Gamma_n(z))_{n\in\mathbb{N}} = \int_0^n e^{-t}t^{z-1}\,dt.$$

We first prove that $z \mapsto \Gamma_n(z)$ is continuous: let $\varepsilon > 0$ and fix $w \in U_1$ and $n \in \mathbb{N}$. Since $z \mapsto t^{z-1}$ is continuous in U_1, there exists $\delta > 0$ such that for every $v \in \mathbb{C}$ such that $|w - v| < \delta$ one has that $|t^{w-1} - t^{v-1}| < \varepsilon/(1 - e^{-n})$ and $v \in U_1$. In this case we can perform the following estimate:

$$|\Gamma_n(w) - \Gamma_n(v)| \leq \int_0^n e^{-t}|t^{w-1} - t^{v-1}|\,dt < (1 - e^{-n})\varepsilon/(1 - e^{-n}) = \varepsilon,$$

proving the continuity of Γ_n in $w \in U_1$ arbitrary, and therefore in all U_1. By Morera's Theorem, we prove Γ_n holomorphic in U_1 by checking that $\int_T \Gamma_n(z)\,dz = 0$ for all triangle $T \subset U_1$. Now, observe that for such a given triangle

$$\int_T \Gamma_n(z)dz = \int_T \int_0^n e^{-t}t^{z-1}\,dt\,dz = \int_0^n \int_T e^{-t}t^{z-1}\,dz\,dt = \int_0^n 0\,dz = 0,$$

since $z \mapsto t^{z-1}$ is holomorphic for all $t > 0$, and we can interchange the integration because both T and $[0, n]$ are compact, and $(t, z) \mapsto e^{-t}t^{z-1}$ is continuous and hence uniformly bounded in $[0, n] \times T$. This shows that $(\Gamma_n)_{n\in\mathbb{N}}$ define a sequence of holomorphic functions on U_1. By taking advantage of Theorem 5.2 of last lecture, to show that Γ is holomorphic in U_1 is suffices to prove that $\Gamma_n \to \Gamma$ uniformly on every compact subset of U_1. Let $K \subset U_1$ be compact, and let $b = \max\{\Re(z-1) : z \in K\} > 0$. Let $N = N(b) > 0$ big enough so that $t/2 \geq b\log(t)$ for all $t > N$. Then, for all $z \in K$ and $n \geq N$ one has that

$$|\Gamma(z) - \Gamma_n(z)| \leq \int_n^{+\infty} e^{-t}e^{\Re(z-1)\log(t)}\,dt \leq \int_n^{+\infty} e^{-t}e^{b\log(t)}\,dt \leq 2e^{-2n},$$

which converges to zero uniformly in K.

(c) Show that $\Gamma(z + 1) = z\Gamma(z)$ for all $z \in U_1$.

SOL: This follows by integration by parts:

$$\Gamma(z + 1) = \int_0^{+\infty} e^{-t}t^z\,dt = [-e^{-t}t^z]_0^{+\infty} + \int_0^{+\infty} e^{-t}zt^{z-1}\,dt = z\Gamma(z).$$

(d) Deduce that Γ allows a unique holomorphic extension to U_0.

SOL: Define the function $\tilde{\Gamma}$ on U_0 by setting

$$\tilde{\Gamma}(z) = \frac{\Gamma(z+1)}{z}, \quad z \in U_0.$$

Since $z \in U_0$ implies $z + 1 \in U_1$ and $z \neq 0$, we deduce that $\tilde{\Gamma}$ is a well defined holomorphic function. On the other side, by the previous point $\tilde{\Gamma}$ coincides with Γ on U_1, showing that it is the unique analytic continuation of Γ from U_1 to U_0.

(e) Deduce that Γ allows a unique holomorphic extension to U.

SOL: We construct the extension on U inductively on $m \in \mathbb{N}_0$ over $U_{-m/2}$ preserving the property $\Gamma(z+1) = z\Gamma(z)$. The case $m = 0$ has been proved in the previous point. Supposing now Γ extended in $U_{-m/2}$, then

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad z \in U_{-(m+1)/2}.$$

defines again an analytic extension, agreeing with the previous one on the set $U_{-m/2}$. The property $\Gamma(z + 1) = z\Gamma(z)$ is ensured by the very definition, an the uniqueness by the properties of analytic functions.