Exercises with a $\star$ are eligible for bonus points.

8.1. Meromorphic functions For $z \in \mathbb{C}$ such that $\sin(z) \neq 0$ define the map

$$\operatorname{cotan}(z) = \frac{\cos(z)}{\sin(z)}.$$

(a) Show that cotan is meromorphic in $\mathbb{C}$, determine its poles and their residues.

SOL: Notice that $\sin(z) = 0$ if and only if $z = k\pi$ for some $k \in \mathbb{Z}$, and therefore cotan is holomorphic in the open domain $\mathbb{C} \setminus \{k\pi : k \in \mathbb{Z}\}$. Since $\{k\pi : k \in \mathbb{Z}\}$ has no accumulation points in $\mathbb{C}$, in order to prove that cotan is meromorphic we are left to show that its singularities are in fact poles. By definition $z = k\pi$ is a pole of cotan if it is a zero of $1/\operatorname{cotan} = \tan$, which is the case since $\cos(k\pi) = (-1)^k$. To compute the residues we notice that all poles have order one since the zeros of tan have order one:

$$\tan(z)' \mid_{z=k\pi} = \left.\frac{1}{\cos^2(z)}\right|_{z=k\pi} = 1 \neq 0.$$

Therefore,

$$\operatorname{res}_{k\pi} \operatorname{cotan} = \lim_{z\to k\pi} (z - k\pi)\frac{\cos(z)}{\sin(z)} = (-1)^k \lim_{z\to k\pi} \frac{(z - k\pi)}{\sin(z)} = (-1)^{2k} = 1,$$

since

$$\lim_{z\to k\pi} \frac{\sin(z)}{z - k\pi} = \lim_{z\to k\pi} \frac{\cos(k\pi)(z - k\pi) + O(|z - k\pi|^2)}{(z - k\pi)} = (-1)^k,$$

by expanding $\sin(z)$ around $k\pi$ at the first order.

(b) Let $w \in \mathbb{C} \setminus \mathbb{Z}$ and define

$$f(z) = \frac{\pi \operatorname{cotan}(\pi z)}{(z + w)^2}.$$

Show that f is meromorphic in $\mathbb{C}$, determine its poles and their residues.

SOL: Since $z \mapsto \operatorname{cotan}(\pi z)$ and $z \mapsto 1/(z + w)^2$ are meromorphic, f is also meromorphic by being the multiplication of the two. Thanks to the previous point, the set of poles of f are $\mathbb{Z} \cup \{-w\}$. The residues at $k \in \mathbb{Z}$ are given by

$$\begin{aligned}\operatorname{res}_k f &= \frac{1}{(k + w)^2} \lim_{z\to k} \frac{\pi(z - k)\cos(\pi z)}{\sin(\pi z)} \\ &= \frac{(-1)^k}{(k + w)^2} \lim_{z\to k} \frac{\pi(z - k)}{\pi \cos(\pi z)(z - k) + O(|z - k|^2)} = \frac{1}{(k + w)^2}.\end{aligned}$$

To compute the order of $-w$ observe that $\operatorname{cotan}(\pi z)$ is equal to zero if and only if $z = k + 1/2$, $k \in \mathbb{Z}$. Hence, if $-w = k + 1/2$, then the pole has order 1 and

$$\begin{aligned} \operatorname{res}_{-w} f &= \lim_{z \to -w} (z+w) f(z) = \lim_{z \to -w} \frac{\pi \cos(\pi z)}{\sin(\pi z)(z+w)} \\ &= \lim_{z \to -w} \frac{\pi(-\pi \sin(-\pi w)(z+w) + O(|z+w|^2))}{\sin(-\pi w)(z+w)} = -\pi^2 = -\frac{\pi^2}{\sin(\pi w)^2}. \end{aligned}$$

If $-w \neq k + 1/2$, then the pole has order 2, and

$$\operatorname{res}_{-w} f = \lim_{z \to -w} \Big((z+w)^2 f(z) \Big)' = \lim_{z \to -w} (\pi \operatorname{cotan}(\pi z))' = -\frac{\pi^2}{\sin^2(\pi w)^2}.$$

(c) Compute for every integer $n \geq 1$ such that $|w| < n$ the line integral

$$\int_{\gamma_n} f \, dz,$$

where γ_n is the circle or radius $n + 1/2$ centered at the origin and positively oriented.

SOL: Observe that γ_n does not intersect with any of the poles of f and contains the pole $-w$. We can therefore apply the Residue Theorem obtaining

$$\int_{\gamma_n} f \, dz = 2\pi i \Big(\operatorname{res}_{-w} f + \sum_{k=-n}^{n} \operatorname{res}_k f \Big) = 2\pi i \Big(-\frac{\pi^2}{\sin^2(\pi w)} + \sum_{k=-n}^{n} \frac{1}{(w+k)^2} \Big).$$

(d) Deduce that

$$\lim_{n \to +\infty} \sum_{k=-n}^{n} \frac{1}{(w+k)^2} = \frac{\pi^2}{\sin(\pi w)^2}.$$

SOL: From the previous point, since

$$\sum_{k=-k}^{k} \frac{1}{(w+k)^2} = \frac{1}{2\pi i} \int_{\gamma_n} f \, dz + \frac{\pi^2}{\sin(\pi w)^2},$$

it suffices to prove that the integral on γ_n vanishes as $n \to +\infty$. Observe that

$$|\operatorname{cotan}(\pi z)| = \left| i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| = \left| \frac{1 + e^{2i\pi z}}{e^{2i\pi z} - 1} \right| \leq \frac{1 + |e^{2i\pi z}|}{||e^{2\pi i z}| - 1|} = \frac{1 + e^{-2\pi \Im(z)}}{|e^{-2\pi \Im(z)} - 1|}.$$

taking advantage of the reverse triangle inequality $|w - u| \geq ||w| - |u||$. Hence, for every $\varepsilon > 0$ the function $\operatorname{cotan}(\pi z)$ is uniformly bounded in the half plane $\{\Im(z) > \varepsilon\}$

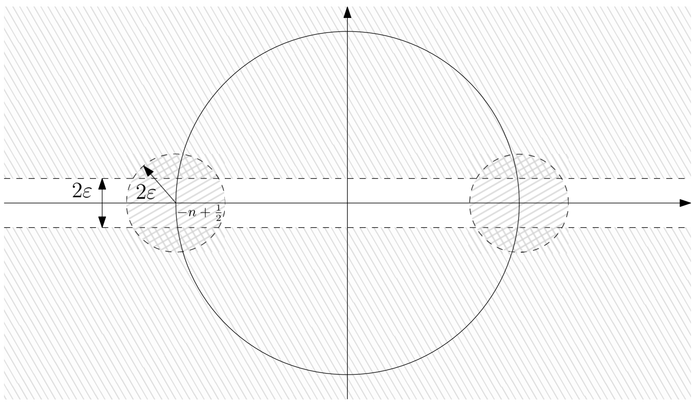
by $C(\varepsilon) = 2/(1 - e^{-2\pi\varepsilon}) > 0$. The same holds true in the half plane $\{\Im(z) < -\varepsilon\}$ since $\text{cotan}(-\pi z) = -\text{cotan}(\pi z)$. Let now $n \in \mathbb{Z}$ and consider a point in a 2ε-neighbourhood of $n\pi + 1/2$, i.e. $u = n + 1/2 + \tau$, for $\tau \in \mathbb{C}$, $|\tau| < 2\varepsilon$. Then, taking advantage of the classical trigonometric identities we can compute

$$\begin{aligned}\text{cotan}(\pi u) = \frac{\cos(u)}{\sin(u)} &= \frac{\cos(\pi(n+1/2))\cos(\pi\tau) - \sin(\pi(n+1/2))\sin(\pi\tau)}{\sin(\pi(n+1/2))\cos(\pi\tau) + \cos(\pi(n+1/2))\sin(\pi\tau)}\\ &= -\frac{\sin(\pi\tau)}{\cos(\pi\tau)} = -\tan(\pi\tau),\end{aligned}$$

whose norm is controlled uniformly in n by some constant $C' = C'(\varepsilon) > 0$ provided $\varepsilon < 1/2$. Hence, fixing $\varepsilon < 1/2$ and covering every circle γ_n with two half planes and two balls centered in the intersection of the real axis we can estimate

$$\begin{aligned}\left|\int_{\gamma_n} f\, dx\right| \leq \int_{\gamma_n} |f|\, dz &\leq \text{length}(\gamma_n)\frac{\pi \max\{C, C'\}}{(n+1/2-|w|)^2}\\ &= \frac{2\pi^2(n+1/2)\max\{C, C'\}}{(n+1/2-|w|)^2} \to 0,\end{aligned}$$

as $n \to +\infty$, as wished.



8.2. Analytic continuation Let $f : \mathbb{C} \to \mathbb{C}$ be and entire function. Then, for every $w \in \mathbb{C}$ we can write

$$f(z) = \sum_{n=0}^{+\infty} a_n^w (z-w)^n,$$

for suitable coefficients $(a_n^w)_n$ in $\mathbb{C}$. Let $B \subset \mathbb{C}$ be an open ball. We suppose that for every $w \in B$ there exists $m \geq 0$ such that $a_m^w = 0$.

(a) For every $n \geq 0$ define the set

$$A(n) := \{w \in B : a_n^w = 0\}.$$

Show that there exists $m \geq 0$ such that $A(m)$ is uncountable.

SOL: By contradiction, suppose that for every n the set $A(n)$ can be written as the image of a complex sequence $(b_{n,k})_k$:

$$A(n) = \{b_{n,k} : k \in \mathbb{N}\}.$$

Then, the set $A = \bigcup_n A(n) = \{b_{n,k} : n, k \geq 0\}$ is also at most countable because it injects into $\mathbb{N} \times \mathbb{N}$. By assumption for every $w \in \Omega$ there exists m such that $w \in A(m)$, implying that

$$B \subset A,$$

which is a contradiction because B is uncountable. Therefore, there must be $m \geq 0$ such that $A(m)$ is uncountable.

(b) Deduce that f is a polynomial or degree at most m.

SOL: Let $m \geq 0$ such that $B \subset \Omega$ is uncountable. For every $w \in B$ one has that

$$f^{(m)}(w) = \sum_{n=m}^{+\infty} \frac{n!}{(n-m)!} a_n^w (z-w)^{n-m} \Big|_{z=w} = a_n^w = 0.$$

But this implies that the holomorphic function $g := f^{(m)}$ has non-isolated zeros in B (the whole set $A(m)$!), which is possible only if $g \equiv 0$ in all B. Hence, f has to be a polynomial of degree at most m in B, and consequently in all $\mathbb{C}$ by analytic continuation.

8.3. Real integrals Compute the following real integrals taking advantage of the Residue Theorem[1].

(a)

$$\int_0^{\pi} \frac{\cos(4t)}{\sin(t)^2 + 1}\, dt.$$

[1]Recall: $\{z_1, \dots, z_N\} \subset \Omega$ poles and $f : \Omega \setminus \{z_1, \dots, z_N\} \to \mathbb{C}$ holomorphic. Then if $\{z_1, \dots, z_N\}$ are inside a simple closed curve γ in Ω, then $\int_\gamma f\, dz = 2\pi i \sum_{j=1}^{N} \text{res}_{z_j}(f)$.

SOL: In order to take advantage of the Residue Theorem, we need to express this real integral as a complex one. We notice that

$$\begin{aligned}\int_0^\pi \frac{\cos(4t)}{\sin(t)^2+1}\,dt &= \Re\Big(\int_0^\pi \frac{e^{4it}}{\sin(t)^2+1}\,dt\Big)\\ &= \Re\Big(\int_0^\pi \frac{e^{4it}}{((e^{it}-e^{-it})/(2i))^2+1}\,dt\Big)\\ &= -4\Re\Big(\int_0^\pi \frac{e^{4it}}{e^{2it}+e^{-2it}-6}\,dt\Big)\\ &= -4\Re\Big(\int_0^\pi \frac{e^{6it}}{e^{4it}-6e^{2it}+1}\,dt\Big).\end{aligned}$$

where in the last line we multiplied numerator and denominator by e^{2it}. We notice now that $t \mapsto e^{2it}$ for $t \in [0,\pi]$ is a parametrization of the unit circle, and hence the above expression is equal to

$$-4\Re\Big(\int_{|z|=1} \frac{z^3}{z^2-6z+1}\frac{1}{2iz}\,dz\Big) = -2\Re\Big(\frac{1}{i}\int_{|z|=1} \frac{z^2}{z^2-6z+1}\,dz\Big)$$

Since the roots of z^2-6z+1 are $z_1 = 3-2\sqrt{2}$ and $z_2 = 3+2\sqrt{2}$, both of order 1, the function $\frac{z^2}{z^2-6z+1}$ has two poles of order one in z_1, z_2. Notice that since only z_1 belongs to the interior of the unit circle, we get

$$\begin{aligned}\int_0^\pi \frac{\cos(4t)}{\sin(t)^2+1}\,dt &= -2\Re\Big(\frac{1}{i}\int_{|z|=1} \frac{z^2}{z^2-6z+1}\,dz\Big)\\ &= -2\Re\Big(2\pi i\frac{1}{i}\operatorname{res}_{3-2\sqrt{2}}\Big(\frac{z^2}{z^2-6z+1}\Big)\Big)\\ &= -4\pi \lim_{z\to 3-2\sqrt{2}} \frac{z^2(z-3+2\sqrt{2})}{z^2-6z+1} = \pi\frac{17-12\sqrt{2}}{\sqrt{2}}.\end{aligned}$$

(b)

$$\int_{-\infty}^\infty \frac{1}{x^4+1}\,dx.$$

SOL: To compute this integral we need to find a suitable contour. For $R > 1$ consider γ_R to be the positively oriented boundary of half disk of radius R: $D := \{z \in \mathbb{C} : \Im(z) > 0, |z| < R\}$. Dividing the curve in the segment $\gamma_R^s := \{-R < t < R\}$ and the upper arc $\gamma_R^a := \{Re^{it} : t \in [0,\pi]\}$, we have that

$$\begin{aligned}\int_{-\infty}^\infty \frac{1}{x^4+1}\,dx &= \lim_{R\to+\infty}\int_{-R}^R \frac{1}{x^4+1}\,dx = \lim_{R\to+\infty}\int_{\gamma_R^s} \frac{1}{z^4+1}\,dz\\ &= \lim_{R\to+\infty}\Big(\int_{\gamma_R} \frac{1}{z^4+1}\,dz - \int_{\gamma_R^a} \frac{1}{z^4+1}\,dz\Big).\end{aligned}$$

We notice that the integral over the arc goes to zero as $R \to +\infty$ since

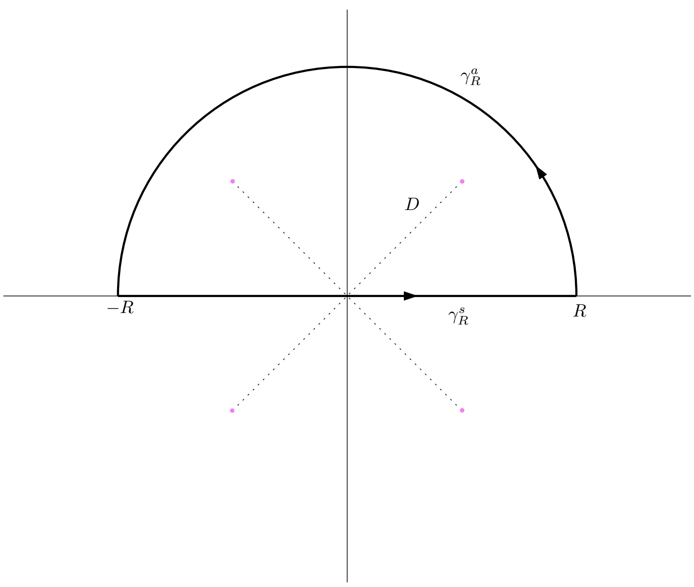
$$\left| \int_{\gamma_R^a} \frac{1}{z^4+1}\, dz \right| \le \frac{\pi R}{R^4 - 1} \to 0, \qquad \text{as } R \to +\infty.$$

On the other hand, the poles of $1/(z^4-1)$ are of order one and equal to $\pm\frac{1+i}{\sqrt{2}}$ and $\pm\frac{1-i}{\sqrt{2}}$. For $R > 1$ only two poles are contained in the the interior of γ: $z_1 = \frac{1+i}{\sqrt{2}}$ and $z_2 = \frac{-1+i}{\sqrt{2}}$. Since

$$\begin{aligned} \operatorname{res}_{z_1} \frac{1}{z^4+1} &= -\frac{1+i}{4\sqrt{2}}, \\ \operatorname{res}_{z_2} \frac{1}{z^4+1} &= -\frac{1-i}{4\sqrt{2}}, \end{aligned}$$

we conclude that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx &= \lim_{R\to+\infty} \left(\int_{\gamma_R} \frac{1}{z^4+1}\, dz - \int_{\gamma_R^a} \frac{1}{z^4+1}\, dz \right) \\ &= \lim_{R\to+\infty} \int_{\gamma_R} \frac{1}{z^4+1}\, dz \\ &= 2\pi i \frac{-2i}{2\sqrt{4}} = \frac{\pi}{\sqrt{2}}. \end{aligned}$$



8.4. Quotient of holomorphic functions Let f, g be two non-constant holomorphic functions on $\mathbb{C}$. Show that if $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$, then there exists $c \in \mathbb{C}$ such that $f(z) = cg(z)$.

SOL: Let $h(z) = \frac{f(z)}{g(z)}$. Since g is not constant, it has isolated zeros, and hence h has isolated singularities. By assumption $|h(z)| \leq 1$ for all z such that $g(z) \neq 0$. In particular, h is bounded in a neighbourhood of the zeros of g, and therefore we extend h to an entire function on the whole complex plane taking advantage of the Riemann continuation Theorem (cf Exercise 5.5). By continuity, the extension h is also uniformly bounded by 1, and therefore by Liouville's Theorem it has to be equal to some constant $c \in \mathbb{C}$. This proves that for all $z \in \mathbb{C}$ such that $g(z) \neq 0$ one has that $f(z) = cg(z)$. If $g(z) = 0$ the assumption $|f(z)| \leq |g(z)| = 0$ concludes the argument: $f(z) = 0 = cg(z)$.