Exercises with a $\star$ are eligible for bonus points.

9.1. Laurent Series A *Laurent series* centered at $z_0 \in \mathbb{C}$ is a series of the form

$$\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

where $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$. We define ρ_0, $\rho_I \in [0,+\infty]$ the *outer* and *inner* radius of convergence as

$$\rho_0 := \left(\limsup_{n\to+\infty} |a_n|^{1/n}\right)^{-1}, \qquad \rho_I := \limsup_{n\to+\infty} |a_{-n}|^{1/n}.$$

If $\rho_I < \rho_0$, we define the *annulus of convergence* as

$$\mathcal{A}(z_0,\rho_I,\rho_0) := \{z \in \mathbb{C} : \rho_I < |z-z_0| < \rho_0\},$$

with the convention $\mathcal{A}(z_0,\rho_I,+\infty) = \{z \in \mathbb{C} : \rho_I < |z-z_0|\}$, so that in particular $\mathcal{A}(z_0,0,+\infty) = \mathbb{C}\setminus\{z_0\}$.

(a) Show that if $\rho_0 > 0$, then the series

$$f_0(z) := \sum_{n=0}^{+\infty} a_n(z-z_0)^n, \quad z \in \mathcal{D}_0(z_0,\rho_0) := \{z \in \mathbb{C} : |z-z_0| < \rho_0\},$$

converges absolutely and uniformly on compact sets. Show that if $\rho_I < +\infty$, then the series

$$f_I(z) := \sum_{n=1}^{+\infty} a_{-n}(z-z_0)^{-n}, \quad z \in \mathcal{D}_I(z_0,\rho_I) := \{z \in \mathbb{C} : \rho_I < |z-z_0|\},$$

converges absolutely and uniformly on compact sets.

SOL: In the case $\rho_0 > 0$, notice that f_0 and ρ_0 coincide with a Taylor expansion in z_0 and the radius of convergence of its associated power series. We know that the series defining f_0 converges absolutely and uniformly on compact subsets of $\mathcal{D}_0(z_0,\rho_0)$ (by Theorem 2.5 in the Lecture Notes). For the case $\rho_I < +\infty$, consider first the power series

$$g_I(\zeta) = \sum_{n=1}^{+\infty} a_{-n}\zeta^n.$$

Then, by the same argument as in the previous case, we know that g_I converges absolutely and uniformly on compact subset in $D(0,1/\rho_I)$, the ball centered at 0 and of radius $(\limsup_{n\to+\infty}|a_{-n}|^{1/n})^{-1} = 1/\rho_I$. Consider the change of variable $\zeta = (z-z_0)^{-1}$. Now, the map $F(z) = (z-z_0)^{-1}$ sends $\mathcal{D}_I(z_0,\rho_I)$ to $D(0,1/\rho_I)\setminus\{0\}$ continuously, and therefore it sends compact subsets of $\mathcal{D}_I(z_0,\rho_I)$ to compact subsets of $D(0,1/\rho_I)\setminus\{0\}$. From the relation $f_I = g_I \circ F$ we deduce that f_I also converges uniformly on compact subsets in $\mathcal{D}_I(z_0,\rho_I)$ as wished.

(b) Show that a Laurent series is divergent for any z satisfying $|z - z_0| > \rho_0$ or $|z - z_0| < \rho_I$.

SOL: The argument is similar to point (a): if $|z - z_0| > \rho_0$ the series $f_0(z)$ diverges, again by Theorem 2.5 in the Lecture Notes. The same hold for $g_I(\zeta)$ when $|\zeta| = |z - z_0|^{-1} > 1/\rho_I$, and hence for $f_I(z)$ when $|z - z_0| < \rho_I$. Since $f = f_0 + f_I$ we conclude that $f(z)$ diverges if $|z - z_0| < \rho_I$ or $|z - z_0| > \rho_0$ as wished.

(c) Deduce that the full Laurent series

$$f(z) := \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$$

defines an analytic function in $\mathcal{A}(z_0, \rho_I, \rho_0)$, and its coefficients are related to f by the formula

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz,$$

for any $n \in \mathbb{Z}$ and $r \in (\rho_I, \rho_0)$.

SOL: Since $f = f_I + f_0$ and f_I is analytic in $\mathcal{D}_I(z_0, \rho_I)$ and f_0 is analytic in $\mathcal{D}_0(z_0, \rho_0)$ by point (a), we deduce that f is analytic in $\mathcal{A}(z_0, \rho_I, \rho_0) = \mathcal{D}_I(z_0, \rho_0) \cap \mathcal{D}_0(z_0, \rho_I)$. Let $r \in (\rho_I, \rho_0)$ and $\varepsilon > 0$ small enough so that $K = \overline{\mathcal{A}(z_0, r - \varepsilon, r + \varepsilon)} \subset \mathcal{A}(z_0, \rho_I, \rho_0)$. Since f converges absolutely an uniformly on the compact set K, we have that

$$\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \sum_{k \in \mathbb{Z}} a_k \frac{1}{2\pi i} \int_{|z-z_0|=r} (z - z_0)^{k-(n+1)}\, dz = a_n,$$

where we exchanged sum and integration by Fubini thanks to the uniform convergence of the series defining f in the compact set K.

9.2. Meromorphic functions Recall the definition of $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

(a) Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Show that f has at most countably many poles.

SOL: Since by definition the poles of a meromorphic function cannot have limit points, any compact subset of $\mathbb{C}$ contains at most finitely many poles. Since every open set Ω in $\mathbb{C}$ is a union of countably many compact sets (for instance, $\mathbb{C} = \bigcup_{n=1}^{+\infty} \{z \in \mathbb{C} : |z| \leq n\}$), it follows that the set of poles of f is at most countable.

(b) Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be meromorphic on $\hat{\mathbb{C}}$. Show that f has at most finitely many poles.

SOL: There exists $R > 0$ such that f is holomorphic for every $|z| > R$. Hence, the poles of f are contained in the compact set $\{|z| \leq R\}$ with the possible exception of ∞. Again, since by definition there is no accumulation point, the number of poles must be finite.

(c) Deduce that if $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic on $\hat{\mathbb{C}}$, than it is a rational function.

SOL: By point (b) we know that the zeros of f in $\mathbb{C}$ are finite, and we can therefore denote them $\{z_1, \dots, z_N\}$ with respective order $\{n_1, \dots, n_N\}$. For each $k \in \{1, \dots, N\}$ we can express f in a neighbourhood of z_k as

$$f(z) = \sum_{n=1}^{n_k} \frac{a_{-n}^k}{(z - z_k)^n} + \sum_{n=0}^{+\infty} a_n^k (z - z_k)^n = f_k(z) + g_k(z),$$

for coefficients $(a_n^k)_{n \geq -n_k}$, where f_k is the principal part of f at z_k, and g_k is holomorphic in a neighbourhood of z_k. Similarly,

$$f(1/z) = f_\infty(z) + g_\infty(z),$$

where g_∞ is holomorphic in a neighbourhood of the origin, and f_∞ is the principal part of $f(1/z)$ at zero. Define now $C(z) = f(z) - f_\infty(1/z) - \sum_{k=1}^N f_k(z)$. Notice that since we removed the principal parts of f at each z_k in the definition of $C(z)$, we deduce that $\{z_1, \dots, z_N\}$ are removable singularities of $C(z)$. The same holds for the possible pole at ∞ since $C(1/z)$ is bounded in a neighbourhood of zero, and therefore $C(z)$ is bounded in $\mathbb{C}$. Hence, by Liouville's Theorem, $C(z) \equiv c \in \mathbb{C}$ is constant, and therefore $f(z) = c + f_\infty(1/z) + \sum_{k=1}^N f_k(z)$ is rational, as claimed.

9.3. Generalization of the Argument Principle Let $\Omega \subset \mathbb{C}$ open, $z_0 \in \Omega$ and $r > 0$ such that $\bar{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\} \subset \Omega$. Suppose that $f : \Omega \to \mathbb{C}$ is homolorphic and that $f(z) \neq 0$ on the circle $\partial D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$. Show that for any holomorphic function $\varphi : \Omega \to \mathbb{C}$ we have that

$$\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f'}{f} \varphi \, dz = \sum_{w \in D(z_0, r) : f(w) = 0} (\operatorname{ord}_w f) \varphi(w).$$

SOL: Let w be a zero of f of order n. Then, there exists g holomorphic and non-vanishing such that $f(z) = (z - w)^n g(z)$. From

$$\frac{f'(z)}{f(z)} \varphi(z) = \frac{n}{z - w} g(z) \varphi(z) + \frac{g'(z)}{g(z)} \varphi(z)$$

we deduce that if $\varphi(w) = 0$, then w is not a zero of $f'\varphi/f$, and hence $\operatorname{ord}_w(f'\varphi/f) = 0 = (\operatorname{ord}_w f)\varphi(w)$. On the other side, if $\varphi(w) \neq 0$, then w is pole of order one of $f'\varphi/f$ with residue

$$\operatorname{res}_w(f'\varphi/f) = \lim_{z \to w} (ng(z)\varphi(z) + (z - w)g'(z)\varphi(z)/g(z)) = ng(w)\varphi(w) = (\operatorname{ord}_w f)\varphi(w).$$

We apply the Residue Theorem to conclude:

$$\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f'}{f}\varphi\,dz = \sum_{w \text{ pole in } |z-z_0|<r} \operatorname{res}_w\left(\frac{f'\varphi}{f}\right) = \sum_{w\in D(z_0,r):f(w)=0} (\operatorname{ord}_w f)\varphi(w).$$

9.4. Application of Rouché Theorem Take advantage of the Rouché Theorem [1] to solve the following.

(a) Show that the polynomial

$$p(z) = z^4 + z^3 + 4z^2 + 1$$

has exactly 2 zeros in $\{z \in \mathbb{C} : 1 < |z| < 3\}$.

SOL: Set $g(z) = z^4 + z^3 + 1$ and $f(z) = 4z^2$ so that $p = f + g$. On $\{z \in \mathbb{C} : |z| = 1\}$ we check that

$$|f(z)| = |4z^2| = 4 > 3 = |z^4| + |z^3| + 1 \geq |z^4 + z^3 + 1| = |g(z)|,$$

so that by the Rouché Theorem, the number of zeros of p in $\{z \in \mathbb{C} : |z| < 1\}$ has to be the same as $4z^2$, hence two. On the other side, choosing now $g(z) = z^3 + 4z^2 + 1$ and $f(z) = z^4$, on $\{z \in \mathbb{C} : |z| = 3\}$ we check that

$$|f(z)| = |z^4| = 81 > 64 = |z^3| + 4|z^2| + 1 \geq |z^3 + 4z^2 + 1| = |g(z)|,$$

proving that p has 4 zeros in $\{z \in \mathbb{C} : |z| < 3\}$. Taking the difference, we conclude that p has exactly $4 - 2 = 2$ zeros in the annulus $\{z \in \mathbb{C} : 1 < |z| < 3\}$.

(b) For every $1 < \lambda$ consider the map

$$f_\lambda(z) := z + \lambda - e^{-z}.$$

Show that f_λ has exactly one zero z_0 in the half plane $\Omega = \{z \in \mathbb{C} : \Re(z) < 0\}$. Show that $\Im(z_0) = 0$, that is z_0 belongs to the real axis.

SOL: Let $R > 1 + \lambda$ and consider the half circle $H_R := \{z \in \mathbb{C} : |z| < R, \Re(z) < 0\}$. Let γ be a counterclockwise parametrization of its boundary

$$\partial H_R = \{it : t \in [-R, R]\} \cup \{Re^{it} : t \in (\pi/2, 3\pi/2)\}.$$

[1] Recall: Let $f, g : \Omega \to \mathbb{C}$ holomorphic and γ a closed, simple curve in Ω such that its interior lies in Ω. If $|f(z)| > |g(z)|$ for all $z \in \gamma$, then f and $f + g$ have the same number of zeros in the interior of γ.

Let $f(z) = z + \lambda$ and $g(z) = -e^z$ so that $f_\lambda = f + g$. For $z \in \{Re^{it} : t \in (\pi/2, 3\pi/2)\}$

$$|f(z)| \geq |z| - \lambda = R - \lambda > 1 \geq |g(z)|$$

and for $z \in \{it : t \in [-R, R]\}$

$$|f(z)| \geq \lambda > 1 = |g(z)|.$$

Since f dominates g on γ, we deduce by Rouché Theorem that f_λ has exactly one zero inside H_R (since f has a unique zero $f(-\lambda) = 0$ of multiplicity one in H_R), and hence in the whole half plane $\{z \in \mathbb{C} : \Re(z) < 0\}$ by arbitrariness of $R > 1 + \lambda$. Now, since f is real valued on $\{z \in \mathbb{C} : \Im(z) = 0\}$, $f(0) = \lambda - 1 > 0$, and $f(-\lambda) = -e^{-\lambda} < 0$, we deduce by continuity that there exists a zero in the segment $(-\lambda, 0) \subset \mathbb{C}$. By uniqueness, this zero coincide with the one whose existence was proved in the first part of the exercise.