Exercises with a $\star$ are eligible for bonus points.

10.1. Laurent Series II Let $0 \leq s_1 < r_1 < r_2 < s_2$, and set $U = \mathcal{A}(0, s_1, s_2)$ and $V = \mathcal{A}(0, r_1, r_2)$ (like in Exercise 9.1). Denote with γ_1 and γ_2 the circles of radius r_1 and r_2, respectively, positively oriented. Let $f : U \to \mathbb{C}$ be a general holomorphic function.

(a) Show that the functions

$$g_1(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w - z}\, dw, \quad \text{for } |z| > r_1,$$

and

$$g_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w - z}\, dw, \quad \text{for } |z| < r_2,$$

are well defined and holomorphic.

SOL: We prove this for g_1 via Morera's Theorem. The proof for g_2 is similar. First, notice that g_1 is continuous in $W = \{z : |z| > r_1\}$: fix $z_1 \in W$ distant $d > 0$ from γ_1. Then, for every $z_2 \in W$ distant $\delta > 0$ from z_1 ($\delta < d/2$) we get

$$\begin{aligned} |g_1(z_1) - g_1(z_2)| &= \left| \frac{1}{2\pi i} \int_{\gamma_1} f(w) \Big(\frac{1}{w - z_1} - \frac{1}{w - z_2} \Big)\, dz \right| \\ &\leq 2 r_1 \max_{w \in \gamma_1} |f(w)| d^{-2} \delta. \end{aligned}$$

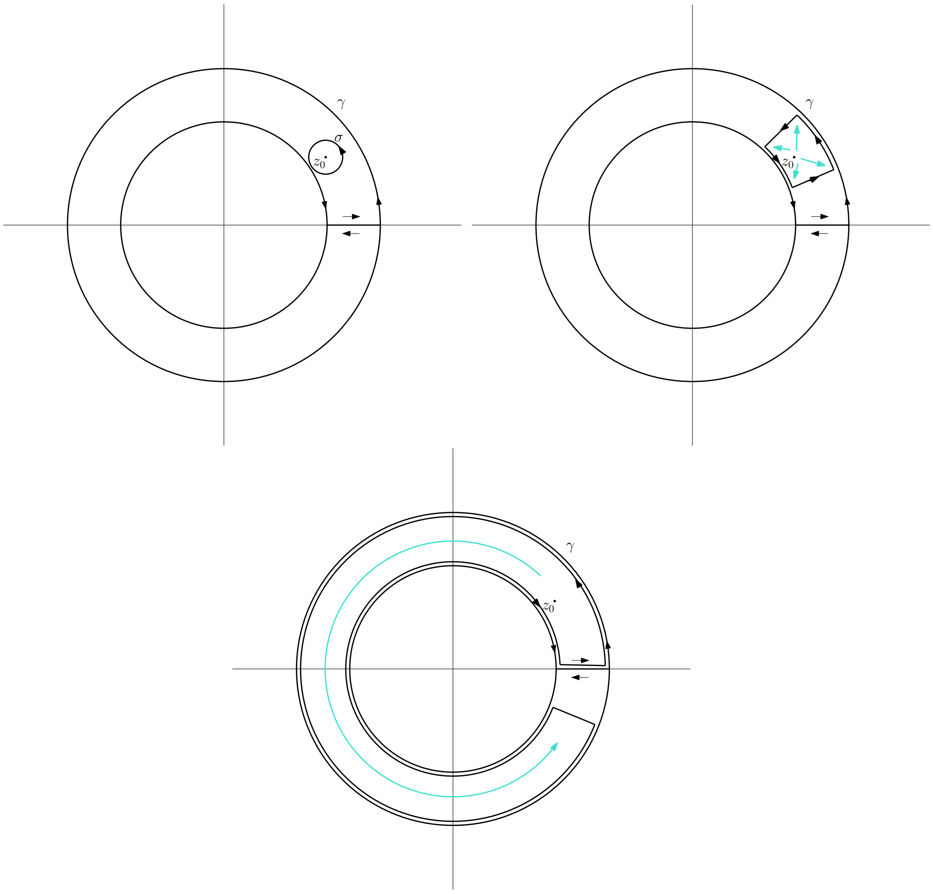
for any $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ small enough. Now given $\varepsilon > 0$ choose $0 < \delta = \delta(\varepsilon) < \min(d/2, \varepsilon d^2 / 2 r_1 \max_{w \in \gamma_1} |f(w)|)$ to conclude the argument. Let now $T \subset W$ a generic triangle in W. Since $z \mapsto 1/(w - z)$ is holomorphic (and hence continuous) in W, by Fubini we check that

$$\int_T g_1(z)\, dz = \frac{1}{2\pi i} \int_T \int_{\gamma_1} \frac{f(w)}{w - z}\, dw\, dz = \frac{1}{2\pi i} \int_{\gamma_1} f(w) \underbrace{\int_T \frac{1}{w - z}\, dz}_{=0 \text{ by Goursat}}\, dw = 0.$$

Hence, g_1 is holomorphic in W by Morera's theorem.

(b) Let γ be the closed curve obtained by going along γ_2 starting at r_2, then along the segment joining r_2 to r_1, then along $-\gamma_1$, and finally back via the segment joining r_1 to r_2. Let $z_0 \in V$ and $r > 0$ small enough such that $\sigma = \{z \in \mathbb{C} : |z - z_0| = r\}$ is in V. Explain why σ and γ are homotopic in U.

SOL: By 'inflating' σ, one can show that it is homotopic to a little sector of annulus. Then, by deforming this sector continuously in the interior of γ it is clear that by overlapping its two flat ends, one obtains the curve γ with the correct orientation.



(c) Show that $f = g_2 - g_1$ in V.

SOL: By independence of Cauchy formula under homotopies, we get that

$$\begin{aligned} f(z) &= \int_\sigma \frac{f(w)}{w-z}\,dw = \int_\gamma \frac{f(w)}{w-z}\,dw \\ &= \int_{\gamma_2} \frac{f(w)}{w-z}\,dw - \int_{\gamma_1} \frac{f(w)}{w-z}\,dw + \int_{r_1}^{r_2} \frac{f(w)}{w-z}\,dw - \int_{r_1}^{r_2} \frac{f(w)}{w-z}\,dw = g_2(z) - g_1(z). \end{aligned}$$

(d) Deduce that f can be represented as a Laurent serie, meaning: there exists a sequence $(a_n)_{n\in\mathbb{Z}}$ such that the series $\sum_{n\geq 1} a_n z^n$ and $\sum_{n\geq 1} a_{-n} z^{-n}$ are absolutely convergent in V, and satisfy

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n, \quad \text{in } V.$$

SOL: By the previous point, it suffices to show that g_1 and g_2 can be represented as a Laurent series. Since g_2 is holomorphic in $\{|z| < r_2\}$ it admits a Taylor expansion (which is in particular a Laurent series) in the disk and $g_2(z) = \sum_{n\geq 0} a_n z^n$. For g_1 we can write

$$\begin{aligned} g_1(z) &= \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw = -\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{z}\frac{1}{1-w/z}\,dw \\ &= -\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{z}\sum_{k\geq 0}\Big(\frac{w}{z}\Big)^k\,dw \\ &= \sum_{n\leq -1}\Big(-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w^{n+1}}\,dw\Big)z^n = \sum_{n\leq -1} a_n z^n, \end{aligned}$$

as wished, where we took advantage of Fubini's Theorem to interchange sum and integration.

10.2. Logarithm Let U be an open and simply connected domain of $\mathbb{C}$, and $f : U \to \mathbb{C}$ a non-vanishing holomorphic function. Fix $z_0 \in U$ and denote with γ_z an arbitrary curve in U connecting z_0 to z.

(a) Show that the function

$$g(z) = \int_{\gamma_z} \frac{f'}{f}\,dw,$$

is well defined and holomorphic in U, and that $g'(z) = \frac{f'(z)}{f(z)}$ for all $z \in U$.

SOL: Since integrating an holomorphic function over a closed curve in a simply connected domain gives always zero, the integral defining g does not depend on the choice of γ_z. Fix $z \in U$ and $\gamma_z : [0,1] \to U$ connecting z_0 to z. Let $\tau \in \mathbb{C}$ with $|\tau|$ small enough so that the curve $\gamma_{z+\tau} : t \mapsto (\gamma_z(t) + t\tau)$ is contained in U. Obviously, $\gamma_{z+\tau}$ connects z_0 with $z+\tau$, and γ_z concatenated with the segment joining z to $z+\tau$ and $-\gamma_{z+\tau}$ is a closed curve. Hence

$$\begin{aligned} \frac{g(z+\tau)-g(z)}{\tau} &= \frac{1}{\tau}\Big(\int_{\gamma_{z+\tau}}\frac{f'}{f}\,dw - \int_{\gamma_z}\frac{f'}{f}\,dw\Big) = \frac{1}{\tau}\int_{\{z+t\tau : t\in[0,1]\}}\frac{f'}{f}\,dw \\ &= \frac{1}{\tau}\int_0^1 \frac{f'(z+t\tau)}{f(z+t\tau)}\tau\,dt = \int_0^1\frac{f'(z+t\tau)}{f(z+t\tau)}\,dt, \end{aligned}$$

which by continuity of f'/f converges to $\int_0^1 f'(z)/f(z)\,dt = f'(z)/f(z)$ as $\tau \to 0$, proving that $g' = f'/f$.

(b) Compute the derivative of $\frac{\exp(g(z))}{f(z)}$.

SOL: By the previous point

$$\Big(\frac{e^g}{f}\Big)' = \frac{e^g g' f - e^g f'}{f^2} = e^g\frac{f'/f\cdot f - f'}{f^2} = 0.$$

(c) Deduce that there exists $\tilde{g}$ holomorphic in U such that $f = \exp(\tilde{g})$. Is this function unique?

SOL: From the previous point we get that e^g/f is equal to some constant $c \in \mathbb{C}$. Therefore, $cf = e^g$, so it suffices to take $c' \in \mathbb{C}$ so that $e^{c'} = c$ and set $\tilde{g} = g - c'$ to have $e^{\tilde{g}} = f$. Notice that the same works by adding to c' an integer multiple of $2\pi i$, so $\tilde{g}$ is not unique in general.

(d) Show that for every $n \in \mathbb{N}$ there exists an holomorphic function $h_n : U \to \mathbb{C}$ such that $(h_n)^n = f$.

SOL: Just take $h_n := \exp(\frac{1}{n}\tilde{g})$, where $\tilde{g}$ is as in the previous point.

10.3. Complex vs Real Is it true that if $u, v : \mathbb{C} \to \mathbb{R}$ are smooth and open maps, then $f = u + iv$ is open? Answer from the perspective of the Open Mapping Theorem.

SOL: No, in general this is false: just consider $u(x, y) = v(x, y) = x$ for instance. Both functions are open since they are projections on the real axis, but the images of $f = u + iv$ are never open because the real axis is not open in $\mathbb{C}$. We deduce that the Open Mapping Theorem is a property of holomoprhic functions which is ensured by the extra condition of Cauchy-Riemann equations.

10.4. Symmetric Rouché

(a) Prove the following variation of Rouché's Theorem by Theodor Estermann (1962): Suppose f, g are holomorphic functions in an open domain $\Omega \subset \mathbb{C}$ and $\gamma \subset \Omega$ a simple, closed curve. If

$$|f(z) + g(z)| < |f(z)| + |g(z)|, \quad \text{for all } z \in \gamma,$$

then f and g share the same number of zeros in the interior of γ.

Hint: consider the convex combination $tf(z) - (1-t)g(z)$.

SOL: Consider the map $h(t, z) := tf(z) - (1-t)g(z)$. Notice that h is continuous in $t \in [0, 1]$, $h(0, z) = -g(z)$ and $h(1, z) = f(z)$. Now, we claim that $h(t, z) \neq 0$ on γ for $t \in (0, 1]$. In fact, if $h(t, w) = 0$ then $f(w) = \frac{1-t}{t} g(w)$ and therefore

$$\begin{aligned} |f(w) + g(w)| &= \left| \left(\frac{1-t}{t} + 1 \right) g(w) \right| = \frac{1}{t} |g(w)| = \left(\frac{1}{t} - 1 \right) |g(w)| + |g(w)| \\ &= |f(w)| + |g(w)|, \end{aligned}$$

contradicting the assumption $|f+g|<|f|+|g|$ on γ. In fact, recall that the triangle inequality $\|a+b\| \le \|a\|+\|b\|$ in $\mathbb{R}^n$ is an equality if and only if a and b are collinear. Now, if suffices to apply the Argument Principle and continuity of h for $t \to 0$:

$$\begin{aligned}\#\{w \in \operatorname{int}(\gamma) : g(w)=0\} &= \int_\gamma \frac{g'}{g}\,dz = \lim_{t\to 0}\int_\gamma \frac{h(t,\cdot)'}{h(t,\cdot)}\,dz \\ &= \lim_{t\to 0}\#\{w\in \operatorname{int}(\gamma) : h(t,w)=0\} = \#\{w\in\operatorname{int}(\gamma) : f(w)=0\},\end{aligned}$$

since the map $t \mapsto \#\{w \in \operatorname{int}(\gamma) : h(t,w)=0\}$ is continuous and has integer value, and hence constant for all $t\in[0,1]$.

(b) Show that the above result implies Rouché Theorem as we have seen it in class.

SOL: Let f and g so that $|g|<|f|$. Apply the Theorem in (a) to $\tilde{g}=g-f$ and $\tilde{f}=f$, observing that $f(z)\neq -g(z)$ on γ.

(c) Show with a simple counterexample that the result of point (a) is stronger than Rouché Theorem as we have seen it in class.

SOL: Take for instance $f=1$ and $g=i$, or f generic and $g=-f$. If you are interested in more sophisticated classes of examples, we refer to Section 1 here: `https://hal.science/hal-01093927/document`

10.5. Maps preserving orthogonality Let $\Omega \in \mathbb{R}^2$ open, and $f:\Omega\to\mathbb{R}^2$ smooth. Show that if f is orientation preserving [1] and sends curves intersecting orthogonally to curves intersecting orthogonally, then f is holomorphic (by identifying $\mathbb{R}^2$ with $\mathbb{C}$).

SOL: By the Cauchy-Riemann equations, it is sufficient to prove that the Jacobian matrix of $f=u+iv$ is pointwise equal to

$$Df(x,y)=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}=\begin{bmatrix}a & b\\ -b & a\end{bmatrix}$$

for some functions a, b. Now, since f sends curves that intersects orthogonally to curves that intersects orthogonally, we get in particular that

$$Df(x,y)\cdot(1,0)^t \perp Df(x,y)\cdot(0,1)^t,$$

that is $(A,C)\perp(B,D)$, implying that $(-C,A)$ is collinear to (B,D), meaning that there exists $\kappa\in\mathbb{R}$ such that $-\kappa C=B$ and $\kappa A=D$. Also, since f preserves the orientation, $0<\det(Df(x,y))=\kappa A^2+\kappa C^2$, implying that $\kappa>0$. We are left to prove that $\kappa=1$. Let now $(x,y)\in\mathbb{R}^2\setminus\{0\}$, then from

$$Df(x,y)\cdot(x,y)^t \perp Df(x,y)\cdot(-y,x)^t \quad\Leftrightarrow\quad (\kappa^2-1)(A^2+C^2)xy=0,$$

implying $\kappa=1$, as wished.

[1] That is the determinant of its Jacobian is positive.