Exercises with a $\star$ are eligible for bonus points.

11.1. Complex integral Evaluate

$$\int_{|z|=1} \frac{z^{10} - 2iz}{2\pi z^{11} + 2z^6 - 3z^4 - i}\, dz.$$

Hint: take advantage of Rouché Theorem and the Homotopy Theorem.

SOL: Let $f(z) = 2\pi z^{11}$ and $g(z) = 2z^6 - 3z^4 - i$. Then, for $|z| = 1$

$$|f(z)| = 2\pi > 6 \geq |2z^6 - 3z^4 - i| = |g(z)|.$$

By Rouché Theorem, $f + g$ has all its eleven zeros inside the unit circle. Therefore, the function that we want to integrate has all its poles in the interior of the unit circle. We can therefore apply the Homotopy Theorem for $\gamma_R = \{z \in \mathbb{C} : |z| = R\} \sim \gamma_1$, $R > 1$, obtaining

$$\begin{aligned}\int_{|z|=1} \frac{z^{10} - 2iz}{2\pi z^{11} + 2z^6 - 3z^4 - i}\, dz &= \int_{|z|=R} \frac{z^{10} - 2iz}{2\pi z^{11} + 2z^6 - 3z^4 - i}\, dz \\ &= \int_0^{2\pi} Rie^{it} \frac{R^{10}e^{10it} - 2iRe^{it}}{2\pi R^{11}e^{11it} + 2R^6e^{6it} - 3R^4e^{4it} - i}\, dt \\ &= \int_0^{2\pi} ie^{it} \frac{e^{10it} - 2iR^{-9}e^{it}}{2\pi e^{11it} + 2R^{-5}e^{6it} - 3R^{-7}e^{4it} - i}\, dt.\end{aligned}$$

But, since

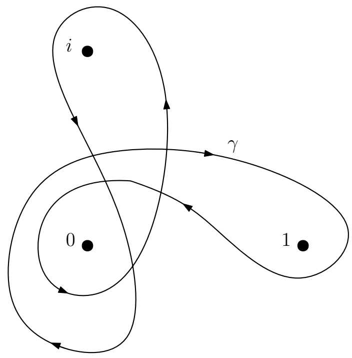
$$\lim_{R\to+\infty} ie^{it} \frac{e^{10it} - 2iR^{-9}e^{it}}{2\pi e^{11it} + 2R^{-5}e^{6it} - 3R^{-7}e^{4it} - i} = (2\pi)^{-1} ie^{it}e^{10it}ie^{-11it} = i(2\pi)^{-1}$$

uniformly in $t \in [0, 2\pi]$, we get by interchanging limit of $R \to +\infty$ and integral that the answer is $2\pi i(2\pi)^{-1} = i$.

11.2. Winding number Evaluate the integral $\int_\gamma f\, dz$ when $f(z) = \frac{e^{iz}}{z^2(z^4-1)}$ and γ is as follows:

SOL: The poles of f are at $z = 0, 1, -1, i, -i$ with multiplicity $2, 1, 1, 1, 1$. The curve γ winds respectively $(-1 + 1), -1, 0, 1, 0$ times around the listed poles. Hence

$$\int_\gamma f\, dz = 2\pi i\Big(-\operatorname{res}_1 f + \operatorname{res}_i f\Big) = 2\pi\Big(-\frac{e^i}{4} - \frac{ie^{-1}}{4}\Big) = -\frac{\pi}{2}(e^i + e^{-1}).$$



11.3. Fractional Residues Prove the following: if z_0 is a simple pole of a meromorphic function f and A_ε is an arc of the circle $\{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$ of an angle $\alpha \in (0, 2\pi]$, then

$$\lim_{\varepsilon \to 0} \int_{A_\varepsilon} f\, dz = \alpha i \operatorname{res}_{z_0}(f).$$

SOL: Since f is meromorphic its poles are isolated, and hence there exists $\varepsilon_0 > 0$ such that z_0 is the unique pole inside $C_{\varepsilon_0} = \{z : |z - z_0| \le \varepsilon_0\}$. Let $0 < \varepsilon < \varepsilon_0$. Since z_0 is simple, we can write $f(z) = \frac{a_{-1}}{z - z_0} + g(z)$ inside C_{ε_0} for some function g holomorphic, where $a_{-1} = \operatorname{res}_{z_0}(f)$. Then, by parametrizing A_ε as $t \mapsto z_0 + \varepsilon e^{it}$, $t \in [t_0, t_0 + \alpha]$, we get that

$$\int_{A_\varepsilon} f\, dz = \int_{A_\varepsilon} \frac{a_{-1}}{z - z_0} + g\, dz = \int_{t_0}^{t_0+\alpha} \frac{a_{-1}}{\varepsilon e^{it}} \varepsilon i e^{it}\, dt + \int_{C_\varepsilon} g\, dz = i\alpha a_{-1} + \int_{A_\epsilon} g\, dz.$$

Notocing now $\left|\int_{A_\varepsilon} g\, dz\right| \le \alpha\varepsilon \max_{|z - z_0| \le \varepsilon_0} |g(z)| = O(\varepsilon)$, we obtain that

$$\lim_{\varepsilon \to 0} \int_{A_\varepsilon} f\, dz = i\alpha a_{-1} + \lim_{\varepsilon \to 0} \int_{A_\varepsilon} g\, dz = i\alpha a_{-1},$$

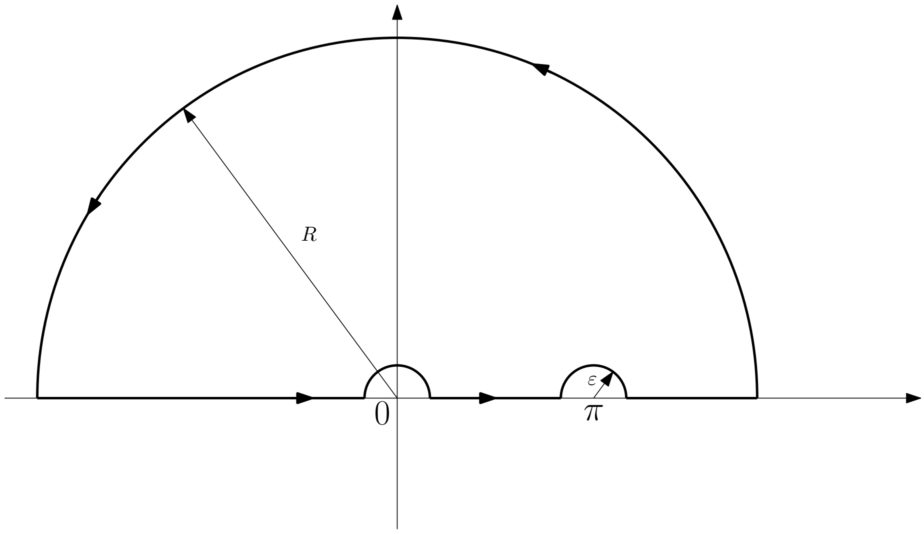
as wished.

11.4. Real integral Evaluate

$$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x(x - \pi)}\, dx.$$

Hint: take a suitable contour in $\mathbb{C}$ that avoids the zeros of the denominator. Take advantage of Exercise 11.3.

SOL: For $R > 2\pi$ and $\varepsilon \in (0, 1)$, consider $\gamma_{\varepsilon,R}$ to be the boundary of the domain $\{z : \varepsilon < |z| < 1, |z - \pi| > \pi, \arg(z) \in [0, \pi]\}$. Consider the function $f(z) = \frac{-ie^{iz}}{z(z-\pi)}$, and



let $A_\varepsilon^1 = \{\varepsilon e^{it} : t \in [0, \pi]\}$, $A_\varepsilon^2 = \{\pi + \varepsilon e^{it} : t \in [0, \pi]\}$, and $C_R = \{Re^{it} : t \in [0, \pi]\}$. Then, since $\int_{\gamma_{\varepsilon,R}} f\,dz = 0$, we get that

$$\int_{-R}^{R} \frac{\sin(x)}{x(x-\pi)}\,dx = \lim_{\varepsilon\to 0}\Big(\int_{A_\varepsilon^1} f\,dz + \int_{A_\varepsilon^2} f\,dz\Big) - \int_{C_R} f\,dz.$$

By Exercise 11.3

$$\lim_{\varepsilon\to 0}\Big(\int_{A_\varepsilon^1} f\,dz + \int_{A_\varepsilon^2} f\,dz\Big) = \pi i(\operatorname{res}_0 f + \operatorname{res}_\pi f) = \pi i(-i/(0-\pi) - ie^{i\pi}/\pi) = -2.$$

On the other side

$$\Big|\int_{C_R} f\,dz\Big| \le \frac{\pi R}{R(R-\pi)} = O(R^{-1}).$$

Hence, we conclude that

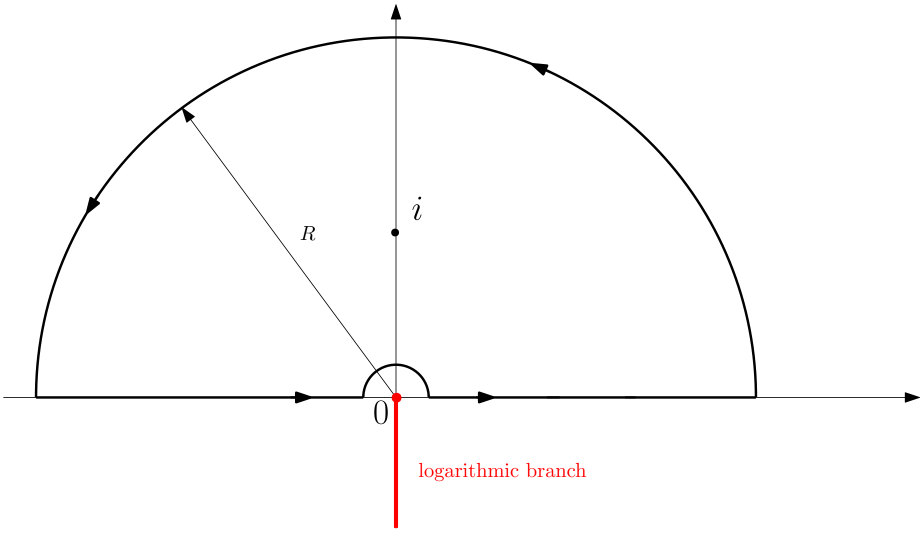
$$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x(x-\pi)}\,dx = \lim_{R\to+\infty}\int_{-R}^{+R} \frac{\sin(x)}{x(x-\pi)}\,dx = -2 - \lim_{R\to+\infty}\int_{C_R} f\,dz = -2.$$

11.5. Real integral II Let $\alpha \in (0,1)$. Evaluate

$$\int_0^{+\infty} \frac{x^{2\alpha-1}}{1+x^2}\,dx,$$

choosing a suitable branch of the logarithm.

SOL: For $R > 1$ and $\varepsilon \in (0,1)$ let $\gamma_{\varepsilon,R}$ be the curve parametrizing the boundary of the domain $\Omega = \{z : \varepsilon < |z| < R, \Im(z) > 0\}$, like in picture. Let $f(z) = \frac{z^{2\alpha-1}}{1+z^2}$. Since



$z^{2\alpha-1} = e^{\log(z)(2\alpha-1)}$, it is convenient to chose the branch of the logarithm to be with argument between $-\pi/2$ and $3\pi/2$, so that the singularity cuts along the negative imaginary axis, and hence does not intersect $\bar{\Omega}$. By the residue Theorem

$$\int_{\gamma_{\varepsilon,R}} f\,dz = 2\pi i\,\mathrm{res}_i\, f = 2\pi i \frac{e^{\log(i)(2\alpha-1)}}{2i} = -\pi i e^{\alpha\pi i}.$$

As $\varepsilon \to 0$ and $R \to +\infty$ the integral of f along $[\varepsilon, R]$ converges to the desired integral. On the other side, the integral over $[-R, -\varepsilon]$ also converges to a multiple of the same value since

$$\int_{[-R,-\varepsilon]} \frac{e^{\log(z)(2\alpha-1)}}{1+z^2}\,dz = -e^{2\alpha\pi i}\int_{[\varepsilon,R]} \frac{e^{\log(w)(2\alpha-1)}}{1+w^2}\,dw,$$

by setting $w = -z$. In fact,

$$\int_{[-R,-\varepsilon]} \frac{e^{\log(z)(2\alpha-1)}}{1+z^2}\,dz = \int_{-R}^{-\varepsilon} \frac{e^{\log(t)(2\alpha-1)}}{1+t^2}\,dt = \int_{\varepsilon}^{R} \frac{e^{\log(-s)(2\alpha-1)}}{1+s^2}\,ds,$$

and since by our choice of the logarithmic branch $\log(-s) = \log(s) + i\pi$, we get

$$\begin{aligned}\int_{[-R,-\varepsilon]} \frac{e^{\log(z)(2\alpha-1)}}{1+z^2}\,dz &= \int_{\varepsilon}^{R} \frac{e^{\log(-s)(2\alpha-1)}}{1+s^2}\,ds = e^{i\pi(2\alpha-1)}\int_{\varepsilon}^{R} \frac{e^{\log(s)(2\alpha-1)}}{1+s^2}\,ds \\ &= -e^{2\alpha\pi i}\int_{[\varepsilon,R]} \frac{e^{\log(w)(2\alpha-1)}}{1+w^2}\,dw.\end{aligned}$$

as wished. Now, observing that

$$|z^{2\alpha-1}| = |z|^{2\alpha-1},$$

the integral over the arc of radius R (that we will call C_R) is of order $O(R^{2\alpha-2})$, and since $\alpha \in (0,1)$ it tends to zero as $R \to +\infty$. In fact:

$$\left|\int_{C_R} f\,dz\right| \leq \int_{C_R} \left|\frac{z^{2\alpha-1}}{1+z^2}\right| dz = \int_{C_R} \frac{|z|^{2\alpha-1}}{|1+z^2|}\,dz \leq \pi R \frac{R^{2\alpha-1}}{R^2-1} = O(R^{2\alpha-2}).$$

For the same reason, the integral over the arc of radius $\varepsilon > 0$ (that we will call c_ε)is of order $O(\varepsilon^{2\alpha})$:

$$\left|\int_{c_\varepsilon} f\,dz\right| \leq \int_{c_\varepsilon} \left|\frac{z^{2\alpha-1}}{1+z^2}\right| dz = \int_{c_\varepsilon} \frac{|z|^{2\alpha-1}}{|1+z^2|}\,dz \leq \pi\varepsilon \frac{\varepsilon^{2\alpha-1}}{1-\varepsilon^2} = O(R^{2\alpha}),$$

and also tends to zero as $\varepsilon \to 0$. We get that

$$(1 - e^{2\alpha\pi i}) \int_0^{+\infty} \frac{x^{2\alpha-1}}{1+x^2}\,dx = -\pi i e^{\alpha\pi i},$$

proving finally that

$$\int_0^{+\infty} \frac{x^{2\alpha-1}}{1+x^2}\,dx = \frac{\pi}{2\sin(\pi\alpha)}.$$