Exercises with a $\star$ are eligible for bonus points.

12.1. Holomorphic injections Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic injection.

(a) Let $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be defined as $g(z) := f(1/z)$. Show that g has no essential singularity at zero. Argue by contradiction taking advantage of the Theorem of Casorati-Weierstrass [1].

SOL: Supposing by contradiction that g has an essential singularity in zero, by the Casorati-Weierstrass Theorem

$$D := g(B(0,1) \setminus \{0\}) = f(\mathbb{C} \setminus \bar{B}(0,1)),$$

is dense in $\mathbb{C}$. Since f is injective, the set $O := f(B(0,1))$ does not intersect D. By the Open Mapping Theorem O is open being image of an open set, and in particular there exists a non-empty open ball $B \subset O$, and hence $B \cap D = \varnothing$. This contradicts the density of D in $\mathbb{C}$, proving that g has no essential singularity in zero.

(b) Taking advantage of the Laurent series of f and g, prove that f is in fact a polynomial.

SOL: From the previous point, g has a pole or a removable singularity in zero. Hence, we can apply Exercise 10.1, that ensures the existence of Laurent series for g and f of the form

$$g(z) = b_{-m}z^{-m} + \cdots + b_{-1}z^{-1} + b_0 + \sum_{k=1}^{+\infty} b_k z^k,$$
$$f(z) = \sum_{k=0}^{+\infty} a_k z^k,$$

where $m \in \mathbb{N} \cup \{0\}$ is the order of the pole of g in zero, and (a_k), (b_k) are suitable coefficients in $\mathbb{C}$. From the relation $g(z) = f(z^{-1})$ we deduce that $a_k = b_{-k}$ for all $k \in \mathbb{Z}$, proving that $a_k = 0$ for all $k > m$, and hence that f is equal to the polynomial expression $f(z) = a_0 + \cdots + a_m z^m$.

(c) Show that f is in the form $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

SOL: Since f has to be injective, $m = 1$. Otherwise, there exist numbers $w \in \mathbb{C}$ such that $f(z) = w$ has exactly $m \geq 2$ solutions, which is a contradiction. We infer that $f(z) = a_0 + a_1 z$, as wanted.

[1] Recall: If $f : B(a,R) \setminus \{a\} \to \mathbb{C}$ holomorphic has an essential singularity in a, then for all $0 < r < R$, $f(B(a,r) \setminus \{a\})$ is dense in $\mathbb{C}$. (Here $B(w,\rho) := \{z \in \mathbb{C} : |z - w| < \rho\}$).

12.2. Pointwise values Suppose h holomorphic in $\mathbb{C}$, $h(0) = 3 + 4i$, and $|h(z)| \leq 5$ if $|z| \leq 1$. What is $h'(0)$ under these conditions?

SOL: We apply the Maximum Modulus Principle to f in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$: since $|h(0)| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5 \geq |f(z)|$ for all $z \in \mathbb{D}$, we see that h attains its maximum modulus in the interior of $\mathbb{D}$, and hence it must be constant, so in particular $h'(0) = 0$.

12.3. Estimates on the modulus Suppose that f is holomorphic on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$, $f(i) = 0$, and that $|f(z)| \leq 1$ for all $z \in \mathbb{H}$. How big can $|f(2i)|$ be under these conditions?

SOL: The map

$$\varphi(z) := i\frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

sends the unit disc $\mathbb{D}$ to the half plane $\mathbb{H}$. Therefore, the map $f \circ \varphi$ sends the unit disc to itself, fixing zero. Now, $\varphi(w) = 2i \Rightarrow w = 1/3$, and hence, from the Schwarz Lemma ($|w| < 1$) one has that

$$|f(2i)| = |f \circ \varphi(w)| \leq |w| = \frac{1}{3},$$

proving that $|f(2i)| \leq 1/3$. Taking $f = \varphi^{-1}$, one can check that in fact this estimate is sharp.

12.4. Schwarz-Pick's Lemma Denote with $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the unit open disk in $\mathbb{C}$, and suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Prove that for all $z \in \mathbb{D}$

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Remark: note that the above expression takes the nicer form: $|f'(a)| \leq \frac{1-|b|^2}{1-|a|^2}$*, for all* $a, b \in \mathbb{D}$ *such that* $f(a) = b$.

SOL: For every $\alpha \in \mathbb{C}$ define the map

$$\psi_\alpha(z) := \frac{z - \alpha}{1 - \bar{\alpha}z}.$$

This map is an automorphism of the disk $\mathbb{D}$, verifying the identity $\psi_\alpha(\alpha) = 0$ for all $\alpha \in \mathbb{C}$. Let $w \in \mathbb{C}$. The map

$$\psi_{f(w)} \circ f \circ \psi_w^{-1}$$

maps 0 to 0, and therefore, by the Schwarz Lemma, for all $z \in \mathbb{D}$ it holds

$$|\psi_{f(w)} \circ f \circ \psi_w^{-1}(z)| \leq |z|.$$

Setting $\tilde{z} = \psi_w(z)$ we get that

$$\left|\frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)}\right| = |\psi_{f(w)} \circ f(z)| = |\psi_{f(w)} \circ f \circ \psi_w^{-1}(\tilde{z})| \leq |\tilde{z}| = |\psi_w(z)| = \left|\frac{z - w}{1 - \bar{w}z}\right|.$$

We can rearrange this inequality as

$$\left|\frac{f(z) - f(w)}{z - w}\right|\left|\frac{1}{1 - \overline{f(w)}f(z)}\right| \leq \frac{1}{1 - \bar{w}z},$$

Letting now $w \to z$, we deduce that the above expression converges to

$$|f'(z)|\left|\frac{1}{1 - |f(z)^2|}\right| = \frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2},$$

where the first identity holds since $f(z) \in \mathbb{D}$ and hence $|f(z)| < 1$.

12.5. Symmetries of the Riemann mapping Let $\Omega \subset \mathbb{C}$ be a non-empty, simply connected domain symmetric with respect to the real axis ($\{\bar{z} : z \in \Omega\} = \Omega$). For $z_0 \in \Omega$ real, denote with $F : \Omega \to \mathbb{D}$ the unique conformal map given by the Riemann Mapping Theorem, so that $F(z_0) = 0$ and $F'(z_0) > 0$. Prove that

$$\overline{F(\bar{z})} = F(z),$$

for all $z \in \Omega$.

Hint: take advantage of Exercise 6.3. on the Schwarz reflection principle.

SOL: Thanks to the proof in Exercise 6.3, and the symmetry of Ω and $\mathbb{D}$ with respect to the real axis, the function $g(z) := \bar{F}(\bar{z})$ is holomorphic in Ω and has image in $\mathbb{D}$. We are left to show that $F = g$. We check that g is biolomorphic: in fact $g^{-1}(w) = \overline{F^{-1}(\bar{w})}$ since

$$g(z) = w \Leftrightarrow \overline{F(\bar{z})} = w \Leftrightarrow F(\bar{z}) = \bar{w} \Leftrightarrow \bar{z} = F^{-1}(\bar{w}) \Leftrightarrow z = \overline{F^{-1}(\bar{w})},$$

and since F^{-1} is holomorphic, it follows again by the proof of Exercise 6.3 that g^{-1} is also holomorphic. Now,

$$g(z_0) = \overline{F(\bar{z}_0)} = \overline{F(z_0)} = \bar{0} = 0 = F(z_0),$$

and

$$g'(z_0) = \lim_{h\to 0} \frac{\overline{F(\overline{z_0 + h})} - \overline{F(\bar{z}_0)}}{h} = \overline{\left(\lim_{h\to 0} \frac{F(z_0 + \bar{h}) - F(z_0)}{h}\right)}$$
$$= \left(\lim_{h\to 0} \frac{F(z_0 + \bar{h}) - F(z_0)}{\bar{h}}\right) = F'(z_0) > 0.$$

By uniqueness of the Riemann Mapping Theorem, we deduce that $F = g$, as wished.