19.9.23

Complex Analysis HS 2023.

Lecturer: Özlem Imamoglu

Coordinator. Lauro Silini

Lectures: Tuesdays 10–12 HG F7
Wed. 8–9 ML D28.

Textbook: Complex Analysis by
E. Stein and R. Shakarchi

Webpage: https://metaphor.ethz.ch/x/2023/hs/401-2303-00L

There will be a bonus system, which can lead upto 0.25 extra points in the final exam.

More information on the webpage.

§ 0. Introduction

Our goal this semester is to study functions $f : \mathbb{C} \to \mathbb{C}$ defined on the complex plane $\mathbb{C}$, or on an open sset of $\mathbb{C}$.

We will see that the study of complex function theory is not simply the study of functions on $\mathbb{R}^2$.

We will see that many ways the theory of functions of one real variable is more more complicated than the theory of of functions of a complex variable.

To give an idea of what I mean let's try to compare and contrast:

(1) It is not too difficult to write down a function of a real variable that has n times differentiable but not infinitely differentiable

$$f(x) := \begin{cases} x^2 \sin 1/x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The derivative of f exists for every x including x=0. $f'(0) = 0$. Hence f is differentiable but its derivative is not continuous. Hence not differentiable twice. By integrating f as many times as you like you can get a function h which is differentiable that many times but not infinitely differentiable.

In contrast: We'll see that if $f : \mathbb{C} \to \mathbb{C}$ is differentiable once then it is differentiable ∞-ly many times.

(2) There are functions $f : \mathbb{R} \to \mathbb{R}$ that are ∞ly many times differentiable whose Taylor series does not represent f ie f is not analytic.

e.g $$f(x) := \begin{cases} \exp\left(\frac{-1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then f is ∞'ly differentiable. Unfortunately at $x=0$ all derivatives are zero. Hence its Taylor series is identically zero and cannot represent f.

In contrast: if $f : \mathbb{C} \to \mathbb{C}$ is a function of a complex variable which is differentiable then f is analytic, i.e. it can be represented by a power series.
(differentiable = analytic)

(3) There are plenty of C^∞ functions of a real variable that are bounded
e.g $\sin x$, $\cos x$

In contrast: We'll see that if $f : \mathbb{C} \to \mathbb{C}$ is differentiable and bounded then it is constant (Liouville's Thm)

(4) For two functions of a real variable f, g
f and g can 'agree' on an open set without being equal

In contrast: If $f, g : \mathbb{C} \to \mathbb{C}$ are 2 differentiable functions which coincides on an arbitrarily small disc (or even on a convergent sequence (z_n)) then $f = g$
(Analytic continuation principle)

∘ ∘ ∘

Remark
The power of complex function theory comes from this "robustness" or rigidity. It is a field where in some sense analysis, geometry, algebra come together.

This we will see allows one to prove theorems that a priori has nothing to do with complex numbers.

Ex ① The Integral.

$$\int_0^\infty \cos(t^2)\,dt = \int_0^\infty \sin(t^2)\,dt = \sqrt{\frac{2\pi}{4}}$$

② Let $\pi(x) := \#\{p \text{ prime} \mid p \leq x\}$

then $\pi(x) \underset{x\to\infty}{\sim} \dfrac{x}{\log x}$ Prime Number Theorem

$\left(\lim_{x\to\infty} \pi(x)/(x/\log x) = 1 \right)$.

(3) If $f \in \mathbb{C}[x]$ a non-zero poly. Then f has a zero in $\mathbb{C}$.
(Fundamental theorem of algebra)
(This is not true for $\mathbb{R}$.)

Before we start with the definition of defferentiability of a complex variable function we recall the definitions and basic properties of complex numbers.

(4) Let $r_4(n) := \#\{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 \mid m_1^2 + m_2^2 + m_3^2 + m_4^2 = n\}$

Then $r_4(n) = 8 \sum\limits_{\substack{d \mid n \\ 4 \nmid d}} d$.

Chapter I

§ 1. Complex numbers and complex plane (Review)

$$\boxed{\mathbb{C} := \{ x + iy \mid x, y \in \mathbb{R},\ i^2 = -1 \}}$$

$\mathbb{R} \hookrightarrow \mathbb{C}$, $\mathbb{R} \subset \mathbb{C}$

$r \longmapsto r + i.0$

For $z = x + iy \in \mathbb{C}$ we define

- Real part of $z := x = \mathrm{Re}(z)$
- Imaginary part of $z := y = \mathrm{Im}(z)$

Complex conjugate of $z := \boxed{x - iy = \bar{z}}$

- $\boxed{\mathrm{Re}(z) = \dfrac{z + \bar{z}}{2}}$
- $\boxed{\mathrm{Im}(z) = \dfrac{z - \bar{z}}{2i}}$
- $z \in \mathbb{R} \Leftrightarrow z = \bar{z}$

  $z \in i\mathbb{R} \Leftrightarrow z = -\bar{z}$ (purely imaginary)

- Complex numbers can also be represented as ordered pairs of real numbers in $\mathbb{R}^2$ $z = (x, y)$ where we have $z = w$ with $w = (u, v)$ if and only if $x = u$ and $y = v$

Addition in $\mathbb{C}$: $z = x + iy$, $w = u + iv$

$z + w := (x + u) + i(v + y)$

as pairs $z + w = (x + u, y + v)$

Multiplication in $\mathbb{C}$

$$z \cdot w = (x+iy)(u+iv)$$
$$= xu + i(xv+yu) + i^2 vy$$
(using $i^2 = -1$)
$$= (xu - vy) + i(xv + yu)$$

as pairs in $\mathbb{R}^2$: $(x,y)\cdot(u,v) := (xu-vy, xv+yu)$

Note $i = (0,1)$, and $(0,1)\cdot(0,1) = (-1,0) = -1$

$\mathbb{R}^2$, together with these 2 operations $+, \cdot$ becomes a field ie

$(\mathbb{R}^2, +, \cdot)$ satisfies ① $(\mathbb{R}^2, +)$ is a comm. group with additive identity $0 := (0,0)$

② $(\mathbb{R}^2 \setminus (0,0), \cdot)$ is a comm group with mult. identity $1 := (1,0)$

③ $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3 = (z_2+z_3)z_1$

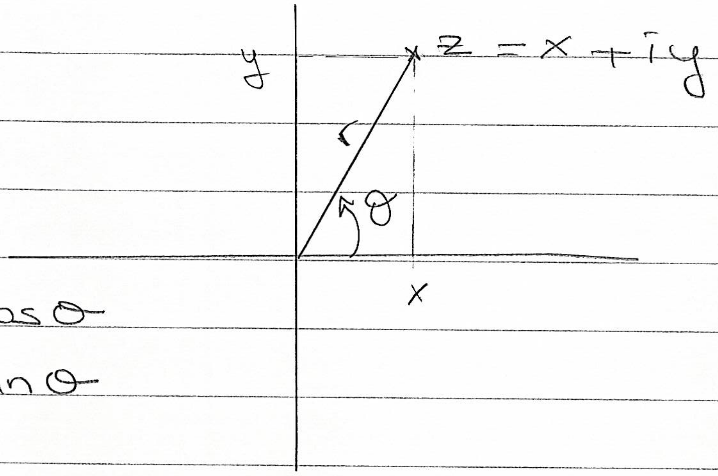
Additive inverse of $z = (x,y)$: $-z = (-x,-y)$

Multip. inverse of z; $0 \neq z = (x,y)$ is $z^{-1} = \frac{\bar{z}}{|z|^2}$

$$= \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

Here $\boxed{|z| := \sqrt{z\bar{z}} = \sqrt{x^2+y^2}}$ is called the norm, or modulus or abs. value of z

We also have the polar coordinate representation of complex numbers



$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$x = r\cos\theta$

$y = r\sin\theta$

where $r \geq 0$, $\theta \in \mathbb{R}$ with $r = |z|$

The polar representation is not unique unless $z \neq 0$ and we restrict $\theta \in (-\pi, \pi]$ (or any other interval of length 2π.

θ is called the argument of z which is defined uniquely up to a multiple of 2π and is denoted by $\arg z$

$$\boxed{\arg z = \{\theta \in \mathbb{R} \mid z = |z|e^{i\theta}\}}$$

The argument of z chosen in the interval $(-\pi, \pi]$ is called the principal argument and denoted by $\operatorname{Arg} z$

$\operatorname{Arg}(i) = \frac{\pi}{2}$ $\quad \operatorname{Arg}(-c) = \pi \quad 0 < c \in \mathbb{R}$

Remark. No assignment of argument is made to $0 \in \mathbb{C}$.

For $z = x + iy \neq 0$ we have

$$\mathrm{Arg}\, z = \begin{cases} \mathrm{Arcsin}(y/|z|) & \text{if } x \geq 0 \\ \pi - \mathrm{Arcsin}(y/|z|) & \text{if } x < 0 \text{ and } y \geq 0 \\ -\pi - \mathrm{Arcsin}(y/|z|) & \text{if } x < 0 \text{ and } y < 0 \end{cases}$$

Here for $t \in [-1, 1]$, $\mathrm{Arcsin}\, t$ is the unique number $u \in [-\pi/2, \pi/2]$ s.t $\sin u = t$

~~Observe that~~ $\arg(z^{-1}) = -\arg z$

$$\arg(zw) = \arg z + \arg w$$

But it is not always the case that

$$\mathrm{Arg}(z^{-1}) = -\mathrm{Arg}\, z \quad \text{or}$$

$$\mathrm{Arg}(zw) = \mathrm{Arg}\, z + \mathrm{Arg}\, w.$$

for example $\mathrm{Arg}(-\frac{1}{2}) = \pi \neq -\mathrm{Arg}(-2)$

and $\pi = \mathrm{Arg}(-1) = \mathrm{Arg}((-i)(-i))$

$$\neq \mathrm{Arg}(-i) + \mathrm{Arg}(-i) = -\frac{\pi}{2} + -\frac{\pi}{2} = -\pi$$

Recall:
- $|z| = 0 \Leftrightarrow z = 0 \quad \forall z \in \mathbb{C}$
- $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2| \quad \forall z_1, z_2 \in \mathbb{C}$
- $|z_1 z_2| = |z_1||z_2| \quad \forall z_1, z_2 \in \mathbb{C}$
- $|\bar{z}| = |z|$
- $|\mathrm{Re}(z)| \leq |z|$, $|\mathrm{Im}(z)| \leq z \quad \forall z \in \mathbb{C}$
- If $z = re^{i\theta}$, $w = se^{i\beta}$ then $zw = rse^{i(\theta+\beta)}$

Next we recall some definitions that we need from topology and Analysis

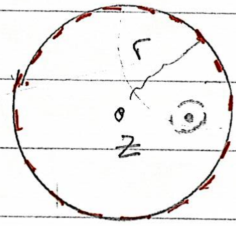
We will denote the open disc of radius r centered at z by $D_r(z)$ or $D(z,r)$

$$D_r(z) := \{ w \in \mathbb{C} \mid |w-z| < r \}$$

$$\overline{D_r}(z) := \{ w \in \mathbb{C} \mid |w-z| \leq r \}$$ is the closed disc.

The boundary of $D_r(z)$ is the circle $C_r(z) = \overline{D_r}(z) \setminus D_r(z)$

$$= \{ w \in \mathbb{C} \mid |z-w| = r \}$$



- A subset $U \subset \mathbb{C}$ is open if $\forall z \in U, \exists r > 0$ such that $D_r(z) \subset U$

  e.g.: $\emptyset$, $\mathbb{C}$, $D_r(z)$, $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

- A subset $U \subset \mathbb{C}$ is called closed if $\mathbb{C} \setminus U$ is open.

  U is closed $\Leftrightarrow$ $\forall$ sequence $(z_n) \in U$, if $z_n \to z$ then $z \in U$

  e.g. $\emptyset$, $\mathbb{C}$, $\overline{D_r}(z)$, $C_r(z)$, $\mathbb{R}$

- A subset $K \subset \mathbb{C}$ is called compact if it is closed and bounded (i.e. $\exists M$ s.t. $|z| < M$ $\forall z \in K$).

(11)

- $K \subset \mathbb{C}$ is compact $\Leftrightarrow$ every sequence $\{(z_n)\} \subset \mathcal{U}$ has a subsequence that converges to a point in $\mathcal{U}$.

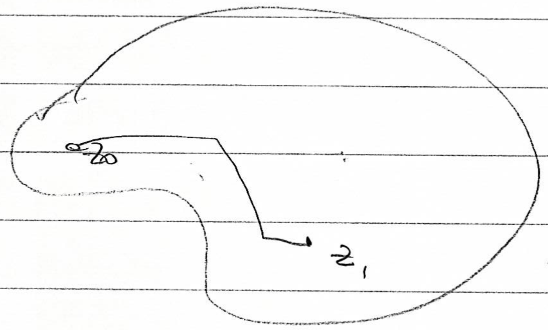
e.g , ϕ, $\overline{D_r(z)}$, $C_r(z)$, $[a,b] \times [c,d]$

An subset A of $\mathbb{C}$ is called <u>disconnected</u> if $\exists$ open sets $\mathcal{U}$ and V such that (i) $\mathcal{U} \cap V = \phi$ (ii) $A \cap \mathcal{U} \neq \phi$, $A \cap V \neq \phi$ and (iii) $A \subset \mathcal{U} \cup V$.

A sset A is called <u>connected</u> if it is not disconnected.

A <u>connected open set</u> $\phi \neq U \subset \mathbb{C}$ is called a <u>region</u> or a <u>domain</u>

- Any pair of distinct points z_0, z_1 in an open connected set $A \subset \mathbb{C}$ can be connected by a polygonal path lying in A



e.g ϕ, $\mathbb{C}$, $D_r(z)$, $\overline{D_r(z)}$, $C_r(z)$, $\mathbb{R}$ are connected

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R} \cup D_1(2i)$ are disconnected.

We recall

① A sequence (z_n) with $z_n = x_n + iy_n$ converges to $z = x + iy$ if one of the following equivalent conditions hold

(i) $x_n \longrightarrow x$ and $y_n \to y$ in $\mathbb{R}$

(ii) $|z_n - z| \longrightarrow 0$ in $\mathbb{R}$

(iii) $\forall \varepsilon > 0, \exists N : \forall m, n \geq N, |z_m - z_n| < \varepsilon$

② Let $U \subset \mathbb{C}$ be an open subset $f : U \longrightarrow \mathbb{C}$ any function.
For $z_0 \in U$ and $w_0 \in \mathbb{C}$ we have

$\lim\limits_{\substack{z \to z_0 \\ z \in U}} f(z) = w_0$ if one of the following equivalent conditions hold

(i) $\forall \varepsilon > 0, \exists \delta > 0, \forall z \in U$ with $|z - z_0| < \delta$, we have $|f(z) - w_0| < \varepsilon$

(ii) If $(z_n) \subset U$ is a sequence with $\lim z_n = z_0$, then $f(z_n)$ converges to w_0.

③ $f : U \to \mathbb{C}$ is continous on U
iff $\forall z_0 \in U \quad \lim\limits_{z \to z_0} f(z) = f(z_0)$
iff $\forall (z_n) \subset U$ with $\lim z_n = z_0$, $\lim f(z_n) = f(\lim z_n) = f(z_0)$