Proof of thm (4.8)'

Clearly (a) $\Rightarrow$ (c). Since then

$\{z \in \Omega \mid f(z) = 0\} = \Omega$.

We'll prove (c) $\Rightarrow$ (b) $\Rightarrow$ (a)

I (c) $\Rightarrow$ (b)

let $Z := \{z \in \Omega \mid f(z) = 0\}$

By assumption Z has a limit point

$a \in \Omega$. let $r > 0$ s.t $D_r(a) \subset \Omega$

(Ω open). f is continuous, and a

is a limit point in Z. ie $\exists\, z_n \in Z \setminus \{a\}$

s.t $\lim z_n = a$. But then

$$0 = \lim f(z_n) = f(\lim z_n) = f(a)$$

Claim $f^{(n)}(a) = 0 \quad \forall n \geq 0$ Hence (b).

Pf of claim: Suppose on the contrary

$\exists\, n > 0$ s.t $f(a) = 0 = \dots f^{(n-1)}(a) = 0$

but $f^{(n)}(a) \neq 0$.

Then as in the proof of the Proposition since f is analytic in $D_r(a) \subset \Omega$ expanding f in a power series there

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k, \qquad |z-a| < r$$

we have that

$$f(z) = (z-a)^n g(z) \quad \text{with } g(a) \neq 0$$

and g is analytic in $D_r(a)$

Since g is continuous, $\exists\ \varepsilon$ $D_\varepsilon(a) \subset D_r(a)$ $(0 < \varepsilon < r)$ s.t $g(z) \neq 0$ on $D_\varepsilon(a)$

Then $f(z) = \underbrace{(z-a)^n}_{\neq 0} \underbrace{g(z)}_{\neq 0\ D_\varepsilon(a)}$

$\forall z \in D_\varepsilon(a) \setminus \{a\}$ i.e $f(z) \neq 0$ in $D_\varepsilon(a) \setminus \{a\}$

Hence $Z \cap (D_\varepsilon(a) \setminus \{a\}) = \phi$

But this says a is not a limit point of Z.

Hence $f^{(n)}(a) = 0 \quad \forall n \geq 0$.

II) $\boxed{(b) \Rightarrow (a)}$ let $A := \{ z \in \Omega \mid f^{(n)}(z) = 0\ \forall n \geq 0 \}$

By assumption $a \in A$, hence $A \neq \phi$.

We'll show that $A = \Omega$, hence $f \equiv 0$.

Recall

for an open Ω, connected means that the only both open and closed sets of Ω are ϕ and Ω.

(It is not possible to find 2 disjoint non-empty open sets Ω_1, Ω_2 s.t. $\Omega = \Omega_1 \cup \Omega_2$

Since $A \neq \phi$, if we can show that A is both open and closed then A will be Ω.

A is open: To see this let $c \in A$

let $r > 0$ s.t $D_r(c) \subset \Omega$. Then

$$f(z) = \sum a_n (z-c)^n, \quad \forall z \in D_r(c)$$

and $a_n = \dfrac{f^{(n)}(c)}{n!} = 0$ since $c \in A$

Hence $f(z) \equiv 0$ in $D_r(c)$

But this means $D_r(c) \subset A$

Hence for an arbitrary $c \in A$, we found a nbhd $D_r(c) \subset A$, which shows A is open.

A is closed: To w.t.s. if $\{z_k\}$ is a sequence of points in A s.t. $\lim\limits_{k \to \infty} z_k = c \in \Omega$

then $c \in A$. To A contains all its limit points.

Let $c \in \Omega$ be a limit point of a sequence $\{z_k\} \subset A$.

Then for any n, and any k

$$f^{(n)}(z_k) = 0 \quad \text{by defn of the set } A$$

But $f^{(n)}$ is continuous. Hence

$$0 = \lim f^{(n)}(z_k) = f^{(n)}(\lim z_k) = f^{(n)}(c)$$

Hence $f^{(n)}(c) = 0 \quad \forall n$, and therefore $c \in A$
Hence A is closed in Ω.

$\boxtimes$

Rmk ① The identity thm makes it clear that the real functions $\sin, \cos, \exp : \mathbb{R} \to \mathbb{R}$ can be uniquely extended to $\mathbb{C}$.

The func'l eqns can also be transfered from reals to complex numbers.

eg. From $\exp(x+y) = \exp(x)\exp(y) \quad \forall x, y \in \mathbb{R}$
we first conclude

$$\exp(z+y) = \exp(z)\exp(y) \quad \forall z \in \mathbb{C}$$

for any fixed $y \in \mathbb{R}$ (but arbitrary y)

And then another application of identity thm gives

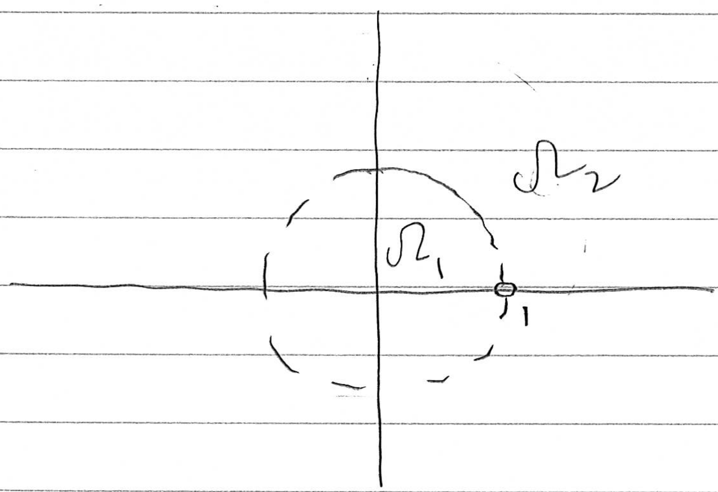
$$\exp(z+w) = \exp(z)\exp(w) \text{ for } z, w \in \mathbb{C}$$

② The geometric series

$$g(z) = \sum_{n=0}^{\infty} z^n \quad , z \in D_1(0)$$

has analytic continuation to $\mathbb{C} \setminus \{1\}$

given by $f(z) = \dfrac{1}{1-z}$



$\Omega_1 = D_1(0)$

$\Omega_2 = \mathbb{C} \setminus \{1\}$

$$g : \Omega_1 \to \mathbb{C} \qquad\qquad f : \Omega_2 \supset \Omega_1 \to \mathbb{C}$$
$$z \to \sum_{n=0}^{\infty} z^n \qquad\qquad z \mapsto \frac{1}{1-z}$$

$f(z) = g(z) = \dfrac{1}{1-z}$ if $z \in D_1(0)$.