**Example** let $f(z) = \sum_{n=0}^{\infty} z^n \quad \forall z \in D_1(0) = \Omega$

$f$ converges for $z \in D_1(0)$ and defines a holom function there.

Note for $z = 1$ $f(z)$ does not converge

Hence for any $\varepsilon$, we cannot define $f(z)$ with $\sum z^n$ on $D_{1+\varepsilon}(0)$ since any such disc contains $z = 1$.

Let $\tilde{\Omega} = \mathbb{C} \setminus \{1\}$ then $\Omega \subset \tilde{\Omega}$

and $F(z) = \frac{1}{1-z}$ is defined on all of $\tilde{\Omega}$ and it agrees with $\sum z^n$ whenever $z \in D_1(0)$.

$F(z) = \frac{1}{1-z}$ is the analytic continuation of $f$ to $\mathbb{C} \setminus \{1\}$.

**Warning!** This does not say that for $z \in \mathbb{C} \setminus \overline{D_1(0)}$, $\sum_{n=0}^{\infty} z^n$ represents $F$.

Note in the identity thm we have 2 holom functions defined on the same set $\Omega$. Here we have

$$f : D_1(0) \to \mathbb{C} \qquad F : \mathbb{C} \setminus \{1\} \to \mathbb{C}$$
$$z \mapsto \sum z^n \qquad z \mapsto \frac{1}{1-z}$$

Remark Not every holom function

$f : \Omega \to \mathbb{C}$ can be extented

to $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$, with $\Omega \subset \tilde{\Omega}$ !

Example . $f(z) = \sum_{n=0}^{\infty} z^{n!}$ converges

for $|z| < 1$ by comparison to the

geometric series, since $|z|^{n!} < |z|^n$.

When we look at series of holom. functions

we'll see that $\sum z^{n!}$ converges abs.

and uniformly on compact ssets of $D_{1-\varepsilon}(0)$
for any $\varepsilon$.

Claim $f$ cannot be extended anywhere
beyond $D_1(0)$

Here is another corollary of the Identity Thm

**Thm** Let $f, g \in \mathcal{H}(\Omega)$, $\Omega$ open connected

If $fg \equiv 0$ then $f \equiv 0$ or $g \equiv 0$.

**Proof** Suppose $f \not\equiv 0$. W.t.s $g \equiv 0$.

$f \not\equiv 0$ so $\exists\, a \in \Omega$ s.t $f(a) \neq 0$

By continuity of $f$

$\exists$ a nbhd of $a$, $D_\varepsilon(a) \subset \Omega$ s.t

$f(z) \neq 0 \quad \forall z \in D_\varepsilon(a)$.

The assumption $f(z)g(z) = 0 \quad \forall z \in \Omega$

then imply that $g(z) = 0 \quad \forall z \in D_\varepsilon(a)$

But then $g\Big|_{D_\varepsilon(a)} = \vartheta\Big|_{D_\varepsilon(a)} = 0$

Using Identity theorem applied to $g: \Omega \to \mathbb{C}$ and the zero function $\vartheta : \Omega \to \mathbb{C}$

gives $g(z) = \vartheta(z) = 0 \quad \forall z \in \Omega$.

**Rmk** The analytic functions on an non-empty open subset $\Omega \subset \mathbb{C}$ form a commutative ring with 1 since the sum and product of holomorphic functions are holomorphic

Recall: $(R, +, \cdot)$ is a ring with 1

① $(R, +)$ is an abelian gp.

② $R$ is a monoid under $\cdot$

ie (i) $(ab)c = a(bc)$ $\forall a, b, c \in R$

(ii) $\exists\ 1 \in R$ such that $a.1 = 1.a$ $\forall a \in R$

③ $\cdot$ distributes over $+$

$a.(b+c) = ab + ac$

$(b+c)a = ba + ca$.

The last thm says that if $\Omega$ is open connected then this ring has no zero divisors!

Our next application is the Morera's theorem, which is a converse to Goursat's theorem.

Recall Goursat's thm says: let $f: \Omega \to \mathbb{C}$ ($\Omega$ open) be a holomorphic function. let $T \subset \Omega$ be a triangle whose interior is also contained in $\Omega$ then

$$\int_T f(z)\,dz = 0.$$

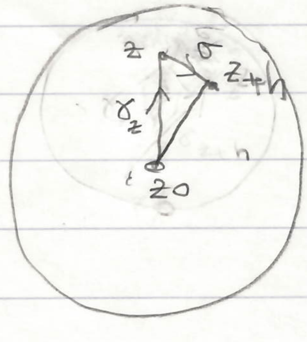
Thm (5.1 II) (Morera's theorem)
let $\Omega \subset \mathbb{C}$ open and $f: \Omega \to \mathbb{C}$ continuous. Assume that for any open disc $D \subset \Omega$ and any triangle $T$ whose inside contained in $D$ we have that

$$\int_T f(z)\,dz = 0.$$

Then $f$ is holom. on $\Omega$.

Proof let $z_0 \in \Omega$, $D_r(z_0) \subset \Omega$

For $z \in D_r(z_0)$ define




$$F(z) := \int_{\gamma_z = [z_0, z]} f(w)\,dw$$

where $\gamma_z : [0,1] \to \mathbb{C}$

$t \to z_0(1-t) + zt$

the line segment joining $z_0$ to $z$

Then for a small $h$ so that $z+h \in D_r(z_0)$,

$$F(z+h) - F(z) = \int_{\sigma = [z, z+h]} f(w)\,dw$$

Since $\int_T f(w)\,dw = 0$ $\forall\, T \subset D_r(z_0)$ by assumption

in particular for $T = \langle z_0, z, z+h \rangle$

Then using continuity of $f$ at $z$ one can show that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

$$F(z+h) - F(z) = \int_{[z, z+h]} \big(f(w) - f(z) + f(z)\big)\,dw$$

$$= f(z) \underbrace{\int_{[z, z+h]} dw}_{h} + \int_{[z, z+h]} \big(f(w) - f(z)\big)\,dw$$

$$\left| \int_{[z,z+h]} \big(f(w)-f(z)\big)\,dw \right| \le \sup_{w\in[z,z+h]} |f(w)-f(z)|\; h$$

so $$\left| \frac{F(z+h)-F(z)}{h} - f(z) \right| \le \sup_{w\in[z,z+h]} |f(w)-f(z)|$$

But $f$ is continuous. Hence

$$\sup_{w\in[z,z+h]} |f(w)-f(z)| \to 0 \quad \text{as } h\to 0$$

and $$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z).$$

So $F$ is holomorphic on $D_r(z_0)$.
but then $F'$ is also holom on $D_r(z_0)$
Since $F'=f$, it follows that
$f$ is holom on $D_r(z_0)$.
But then $f$ is holom on all of
$\Omega$ as $z_0\in\Omega$ was arbitrary.

## § 5.2 Sequences of holomorphic functions

It is known from real analysis that pointwise convergence of a sequence of functions lead to pathologies, such as ~~the~~ the pointwise limit of a sequence of continuous functions is not necessarily continuous.

To avoid this we used a stronger form of convergence; uniform convergence. For example The limit of a uniformly convergent sequence of continuous functions is continuous.

We also have that uniformly conv. seq. of integrable functions converges to an integrable function.
Hence uniform convergence of sequence of functions has better stability properties
But uniformly convergent seq. of differentiable functions does not necessarily have differentiable limits.

We'll see that sequence of complex functions have much better stability properties
As in the real case, uniform limit of a sequence of continuous functions is continuous and similarly line integrals of a uniform. conv. sequence of functions converge to the line integral of the limit function

In contrast to the situation in real analysis we'll see complex differentiability is also stable with respect to uniform convergence

Recall: A sequence $f_1, f_2, \ldots : \Omega \to \mathbb{C}$ of functions defined on an open set $\Omega \subseteq \mathbb{C}$ is called uniformly convergent (in $\Omega$) to the limit $f : \Omega \to \mathbb{C}$
if
$\forall \varepsilon > 0$, $\exists N > 0$ s.t

$$|f(z) - f_n(z)| < \varepsilon \quad \forall n \geq N, \forall z \in \Omega$$

$$\Longleftrightarrow \lim_{n \to \infty} \sup \{ |f(z) - f_n(z)| : z \in \Omega \} = 0.$$

( $N$ does not depend on $z$, only on $\varepsilon$ )

In fact we only need uniform convergence locally, or equivalently uniform convergence on compact subsets.

**Defn** let $\Omega \subset \mathbb{C}$ be open. $f_n : \Omega \to \mathbb{C}$ a sequence of functions. $(f_n)_{n\geq 1}$ is called locally uniformly convergent or compactly convergent or uniformly convergent on compact sets if the following equivalent conditions are satisfied

① $\forall a \in \Omega \ \exists \varepsilon > 0$ s.t $B_\varepsilon(a) \subset \Omega$ s.t $\left(f_n|_{B_\varepsilon(a)}\right)$ converges uniformly

② For every compact set $K \subset \Omega$ $(f_n|_K)$ converges uniformly

Note ① $\Rightarrow$ ② Since $K$ is covered by finitely many discs in ①

② $\Rightarrow$ ① Since $\Omega$ is open, $\forall a \in \Omega$ there is a closed disc, (ie compact) $a \in D \subset \Omega$.

Remark Note that since continuity is a local property even in the case of real valued functions, local uniform convergence of continuous functions will imply continuity of the limit function

Hence similar to the real case one can show

**Prop.** $(f_n)_{n\geq 1}$, $f_n : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$ open
$f_n$ continuous
If $(f_n)$ converges uniformly on compact sets to $f$ then $f$ is continuous.

The main theorem we have is

**Thm 5.2** Let $(f_n)_{n\geq 1}$ be a sequence of holomorphic functions on $\Omega$, $\Omega \subseteq \mathbb{C}$ open. If $(f_n)$ converges uniformly to a function $f$ in every compact set of $\Omega$. Then $f$ is also holomorphic.

**Proof.** Since $f_n$ are each holom, they're also continuous. Hence by above Prop. their limit $f$ is also continuous

To show $f$ is also holomorphic we'll use Morera's theorem, and the fact that any triangle $T$ is compact.

By Morera's thm, since $f$ is continuous to show $f$ is holom, it is enough to show $\int_T f(w)\,dw = 0$ for any open disc $D$, $T \subset D \subset \Omega$ and $T$ triangle contained in $D$.

Let $D = D_r(z_0) \subset \Omega$ an open disc in $\Omega$
T any triangle with inside contained in D.

By Goursat's thm $\int_T f_n(w)\,dw = 0 \quad \forall n \geq 1$

Since $f_n(z) \to f(z)$ uniformly on compact sets

and T is compact
$f_n(z) \to f(z)$ uniformly $\forall z \in T$

$$\left| \int_T f_n(z)\,dz - \int_T f(z)\,dz \right|$$

$$\leq \int_T |f_n(z) - f(z)|\,|dz|$$

$$\leq \underbrace{\sup_{z \in T} |f_n(z) - f(z)|}_{\downarrow \atop 0} \text{ (length of T)}$$

0 since $f_n(z) \to f(z)$ unif on T

Hence $\underbrace{\lim \int_T f_n(z)\,dz}_{0} = \int_T f(z)\,dz$

Hence $\int_T f(z)\,dz = 0 \Rightarrow f$ is holom on $\Omega$ ☒