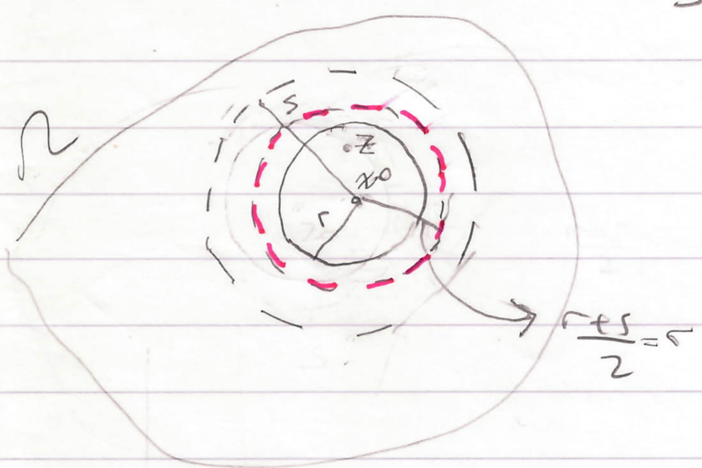
25.10.23

We have the following generalization

Thm 5.3 $\{f_n\}_{n=1}^{\infty}$ a seq. of holomorphic functions in $\Omega \subset \mathbb{C}$ (Ω open) such that $f_n \to f$ unif. on every compact sset of Ω, then $\{f_n'\}_{n\geq 1}$ converges uniformly to f' on every compact sset.

Proof. Let $z_0 \in \Omega$, $r>0$ s.t $\overline{D_r(z_0)} \subset \Omega$. f_n converges unf to f on $\overline{D_r(z_0)}$

Let $s>r$ s.t $D_s(z_0) \subset \Omega$



Let $\sigma = \frac{r+s}{2} \in (r, s)$

we then have

By CIF for derivatives II (Cor 4.2)

$$f'(z) = \frac{1}{2\pi i}\int_{C_\sigma(z_0)} \frac{f(w)}{(w-z)^2}\,dw$$

and $$f_n'(z) = \frac{1}{2\pi i}\int_{C_\sigma(z_0)} \frac{f_n(w)}{(w-z)^2}\,dw$$

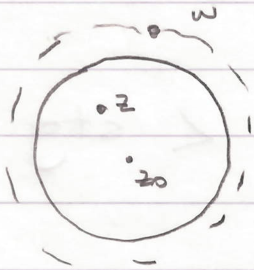
for every $z \in \overline{D_r(z_0)} \subset D_\sigma(z_0)$

Hence for $z \in \overline{D_r(z_0)}$ we have

$$|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i} \int_{C_\sigma(z_0)} \frac{f_n(w) - f(w)}{(w-z)^2}\, dw \right|$$

$$\leq \frac{1}{2\pi}(2\pi\sigma) \sup_{w \in C_\sigma(z_0)} \left| \frac{f_n(w) - f(w)}{(w-z)^2} \right|$$

But for $w \in C_\sigma(z_0)$ and $z \in \overline{D_r(z_0)}$, $|w - z_0| = \sigma$, $|z - z_0| < r$



$$|w - z| = |w - z_0 - (z - z_0)|$$

$$\geq \big| |w - z_0| - |z - z_0| \big|$$

$$\sigma - r$$

Hence $|f_n'(z) - f'(z)| \leq \frac{\sigma}{(\sigma - r)^2} \sup_{C_\sigma(z_0)} |f_n(w) - f(w)|$

Since $f_n(w) \to f(w)$ unif. on the compact set $C_\sigma(z_0)$

we have that $f_n' \to f'$ unif on $\overline{D_r(z_0)}$

Since every compact set is contained in a union of finitely many such discs we're done

$\boxtimes$

Remarks

① These theorems are often used to prove holomorphicity of functions defined by infinite series. Let f_n be a seq. of holom. functions.

If $F(z) = \sum_{n=1}^{\infty} f_n(z) \qquad z \in \Omega$

Let $S_N(z) = \sum_{n=1}^{N} f_n(z)$. Then $S_N(z)$ is holom.

if $\{S_N(z)\}_{N=1}^{\infty}$ converges uniformly on compact ssets of Ω

then $\lim S_N(z) = F(z)$ is also holomorphic

② For series of functions, we have also the following useful theorem of Weierstrass; called Weierstrass M-test

Thm Let $f_n : \Omega \to \mathbb{C}$ a sequence of functions $U \subset \Omega$ a non-empty sset.
Suppose $\exists$ a sequence of real numbers $M_n \geq 0$ s.t

$|f_n(z)| \leq M_n \ \forall n \in \mathbb{N}, \forall z \in U$ and $\sum_{n=0}^{\infty} M_n < \infty$

Then $\sum_{n=1}^{\infty} f_n$ converges abs. and unif on U

Proof Exercise.

<u>Example</u> For $s \in \mathbb{C}$, $s = \sigma + it$
$\sigma, t \in \mathbb{R}$, $n \in \mathbb{N}$

the function $s \longmapsto n^s := \exp(s \log n)$

is an analytic function on $\mathbb{C}$

$$|n^s| = \left| e^{(\sigma + it)\log n} \right| = e^{\sigma \log n} = n^\sigma$$

Then we have

Proposition (2.1. Chapt 6) The series $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely and uniformly on every half plane

$$U_\delta := \{ s \in \mathbb{C} \mid \operatorname{Re} s \geq 1 + \delta \}, \quad \delta > 0$$

and is holomorphic in $\{ s \in \mathbb{C} \mid \operatorname{Re} s > 1 \}$.

<u>Proof</u>- For each $\delta > 0$ we have
If $\operatorname{Re} s = \sigma \geq 1 + \delta > 1$
Then the series $\zeta(s)$ is uniformly bounded

by $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty$ since

$$|\zeta(s)| = \left| \sum_{n=1}^{\infty} \frac{1}{n^s} \right| \leq \sum \frac{1}{|n^s|} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re} s}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly on every half plane $\mathrm{Re}\, s \geq 1+\delta > 1$, $\forall \delta$ and hence defines a holomorphic function in $\mathrm{Re}\, s > 1$

(Every compact sset of $\{s \mid \mathrm{Re}\, s > 1\}$ is contained in such a half plane $\mathrm{Re}\, s \geq 1+\delta$)

Example For $z \in \mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\} \subset \mathbb{C}$
(Note $\mathbb{H}$ is open)

we define the theta function

$$\theta(z) := \sum_{n=0}^{\infty} e^{2\pi i n^2 z}$$

Prop. $\forall z \in \mathbb{H}$, $\theta(z)$ is well-defined ie the series converges and defines a holomorphic function there

Proof: We'll show that it converges uniformly on any subset of the form $\mathbb{H}_\delta := \{z \in \mathbb{C} \mid \mathrm{Im}\, z \geq \delta\}$ with $\delta > 0$ since any compact sset of $\mathbb{H}$ is contained in such a set, this will imply the result

For $z \in \mathbb{H}_\delta$, $z = x + iy$, $y \geq \delta > 0$.

$$\left|e^{2\pi i n^2 z}\right| = \left|e^{2\pi i n^2 x}\right| \cdot \left|e^{-2\pi n^2 y}\right|$$

$$= e^{-2\pi n^2 y} \leq e^{-2\pi n y} \quad \forall n.$$

Since $y \geq \delta$, $e^{-2\pi n y} \leq e^{-2\pi n \delta} < 1$

Hence $\left|\sum_{n=0}^{\infty} e^{2\pi i n^2 z}\right| \leq \underbrace{\sum_n e^{-2\pi n \delta}}_{\text{geometric series}} < \infty$

Hence $\sum_{n=0}^{\infty} e^{2\pi i n^2 z}$ converges uniformly on $\mathbb{H}_\delta$ for any $\delta > 0$

Hence it defines a holomorphic function on $\mathbb{H}$

$\boxtimes$

Remark We'll come back to $\zeta(s)$ and $\theta(z)$, and use $\theta(z)$ to show that $\zeta(s)$ (which is defined by the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\operatorname{Re} s > 1$) has an analytic continuation to $\mathbb{C} \setminus \{1\}$