31-10-23

Finally we also a similar theorem for functions defined in terms of integrals ie similar to the thms for functions defined in terms of infinite series.

Many special functions in mathematics are defined in terms of integrals of the type

$$f(z) := \int_a^b F(z,t)\,dt$$

or as limits of such integrals

For example: $\Gamma(z) := \lim_{\varepsilon \to \infty} \int_{1/\varepsilon}^{\varepsilon} e^{-t}\, t^z \frac{dt}{t}$

We have the following Thm

Thm 5.4 Let $\Omega \subset \mathbb{C}$ open, $I = [a,b] \in \mathbb{R}$ a closed bounded interval

Let $F : \Omega \times I \longrightarrow \mathbb{C}$ be a function with the following properties

(a) $F : \Omega \times I \longrightarrow \mathbb{C}$ is continuous on $\Omega \times I$

(b) For each $t_0 \in I$, the function $f_{t_0}(z) := F(z, t_0) : \Omega \to \mathbb{C}$ is holomorphic.

Then the function $f(z)$ defined by the integral

$$f(z) := \int_a^b F(z,t)\,dt$$ is

holomorphic on Ω.

Proof The idea is to use the Riemann sums to approximate the integral: let

$$f_n(z) := \left(\frac{b-a}{n}\right) \sum_{j=0}^{n-1} F\left(z, a + \frac{b-a}{n} j\right)$$

Then $f_n(z)$ is a finite sum of holom. functions hence holomorphic.

We want to show that $f_n(z)$ converges to f uniformly on compact ssets. Then using Thm 5.3 we can conclude that f is holom.

Let $K \subset \Omega$ be compact.

We use that a continuous function $F : \Omega \times I \to \mathbb{C}$ when restricted to the compact set $K \times I$ is unif continuous

Hence $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall (z_i, t_i) \in K \times I$
$i = 1, 2$

if $|z_1 - z_2| < \delta$ and $|t_1 - t_2| < \delta$ then

$$\left| F(z_1, t_1) - F(z_2, t_2) \right| < \frac{\varepsilon}{b-a}.$$

Let n be large enough so that $\frac{b-a}{n} < \delta$

Then $\forall z \in K$

$$f_n(z) - f(z) = \sum_{j=0}^{n-1} \int_{a + j\left(\frac{b-a}{n}\right)}^{a + (j+1)\frac{b-a}{n}} \left[F\left(z, a + j\left(\frac{b-a}{n}\right)\right) - F(z,t) \right] dt$$

Using $f(z) = \int_a^b F(z,t)\,dt = \int_a^{a + \left(\frac{b-a}{n}\right)} F + \int_{a + \frac{b-a}{n}}^{a + 2\left(\frac{b-a}{n}\right)} F + \cdots \int_{a + (n-1)\frac{b-a}{n}}^{b} F$

and $f_n(z) = \frac{(b-a)}{n} \sum_{j=0}^{n-1} F\left(z, a + j\left(\frac{b-a}{n}\right)\right)$

$$= \sum_{j=0}^{n-1} \underbrace{\int_{a + j\left(\frac{b-a}{n}\right)}^{a + (j+1)\left(\frac{b-a}{n}\right)} F\left(z, a + j\left(\frac{b-a}{n}\right)\right) dt}_{= F\left(z, a + j\left(\frac{b-a}{n}\right)\right) \cdot \left(\frac{b-a}{n}\right)} =$$

since the integrand is indep of t

Now since $t \in \left[a + j\left(\frac{b-a}{n}\right), a + (j+1)\frac{b-a}{n} \right]$

$$\left| t - \left(a + j\left(\frac{b-a}{n}\right)\right)\right| < \frac{b-a}{n} < \delta$$

Since z-arguments are equal we also have

$0 = |z - z| < \delta$

Hence $\left| F\left(z, a + j\left(\frac{b-a}{n}\right)\right) - F(z,t)\right| < \frac{\varepsilon}{b-a}$

and $\left| f_n(z) - f(z) \right| \leq \frac{\varepsilon}{b-a} \sum_{j=0}^{n-1} \left(\frac{b-a}{n}\right) = \varepsilon$

$\forall\ z \in K$ which gives the uniform convergence of f_n to f on K.

Hence f is holomorphic ■

<u>Remark</u> One can with some more work can also show that $f'(z)$ is given by

$$f'(z) = \int_a^b \underbrace{F'(z,t)}_{f_t'(z)}\,dt \qquad \forall z \in \Omega.$$

i.e. we can interchange $\int$ and $\frac{\partial}{\partial z}$

Remark Many special functions that appear as solns of differential equations, for ex. Bessel functions have integral representations

eg: $J_n(z)$ is defined as solution of Bessel's diff equations

$$z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - n^2) f = 0$$

For $n \in \mathbb{Z}$,

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin t} \cdot e^{-int}\, dt$$

$$J_0(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin t}\, dt$$

$F(z,t) = e^{iz\sin t}$ is continuous on $\mathbb{C} \times \mathbb{R}$

For each $t \in [-\pi, \pi]$

$f_t(z) = e^{iz\sin t} : \mathbb{C} \longrightarrow \mathbb{C}$

is holomorphic

Hence the function $\int_{-\pi}^{\pi} e^{iz\sin t}\, dt$ is holom. on $\mathbb{C}$.

$$J_0'(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin t}\,(i\sin t)\, dt$$

III. Meromorphic functions and Residue Formula

Goal: To extend Cauchy's theorem and C.I.F from holomorphic functions to functions which might have singularities.

Recall: Cauchy's thm, $\int_\gamma f\,dz = 0$

for any γ closed curve, $f \in \mathcal{H}(\Omega)$
γ and its interior is contained in Ω.

CIF: $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w - z}$

$\forall z \in D$, a disc, $\partial D = C$ and f is holom in D.

To this end we'll first look at isolated singularities of a function f.

We'll see there are 3 prototypes: $\frac{\sin z}{z}$, $\frac{1}{z}$, $e^{1/z}$

$\frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$ shows that $z=0$ is a "removable" singularity

One can view the RHS as an anal. cont. to all $\mathbb{C}$ of LHS $= \frac{\sin z}{z}$

$$\left|\frac{1}{z}\right| \to \infty \quad \text{as} \quad z \to 0$$

where as $|e^{1/z}|$ oscillates. For example
if $z \to 0$ on positive real numbers
then $|e^{1/x}| \to \infty$ as $x \to 0$, $(x > 0)$
if $z \to 0$ on negative real numbers
then $|e^{1/x}| \to 0$ as $x \to 0$ $(x < 0)$

These 3 examples of singularities are what we call removable, a pole and an essential singularity resp.

2 We'll prove a generalization of Cauchy's thm to functions that are holom except for finitely many isolated points. This will lead us to the

Residue formula: If f is holom in an open set Ω containing a circle and its interior except for finitely many points $z_1, \dots, z_n$ insite C then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \underbrace{\mathrm{Res}_{z_k} f}_{a_{-1}}$$

where we'll also see that f in a nbhd of z_0 has the form

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z-z_0} + G(z), \text{ where } G \in \mathcal{H}(D_r(z_0))$$

This thm, like Cauchy's thm can be used to evaluate many real integrals and complex line integrals.

It will also lead to many theoretical results just like Cauchy's thm did

<u>Argument principle</u> = which allows us to count the number of zeroes (and poles) of holomorphic (meromorphic) functions insite closed curves.

<u>Rouché's thm</u>: A holom. function can be perturbed slightly without changing the number of its zeros.
f, g holom in an open set containing a circle C and its interior.
if $|f(z)| > |g(z)| \quad \forall z \in C$ then f and $f+g$ have the same # of zeroes insite C.

<u>Open mapping thm</u>: f is holom, non-constant in an open connected region Ω then f is an open map ie image of an open set is open

<u>Maximum modulus principle</u> If f is non-constant

on Ω open connected, compact closure $\overline{\Omega}$
if f is cont. on $\overline{\Omega}$ then

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \overline{\Omega} - \Omega} |f(z)|$$

Another way to say this

$$\underbrace{\max_{z \in \overline{\Omega}}}_{} |f(z)| = \max_{z \in \overline{\Omega}-\Omega} |f(z)|$$

exists because
$\overline{\Omega}$ is compact

We start with definitions of singularities.

Defn let $z_0 \in \mathbb{C}$. z_0 is called a (possible) isolated singularity of f if $\exists\, r>0$ such that f is holomorphic in the punctured disc $D_r(z_0) \setminus \{z_0\} =: D_r^*(z_0)$

(or if $\exists$ U open, $z_0 \in U \subset \Omega$ s.t $f \in \mathcal{H}(U - \{z_0\})$)

eg. $f = \mathbb{C} \setminus \{0\} \to \mathbb{C}$
$z \to z$

$z=0$ is an isolated singularity of f because f is not defined there, but f can in fact be extended to all $\mathbb{C}$ by defining $f(0)=0$. In this case

$z=0$ is a "removable singularity"

On the other hand $f: \mathbb{C}\setminus\{0\} \to \mathbb{C}$

$z \longmapsto 1/z$

has a singularity at $z=0$, which cannot be removed.

Defn

An isolated singularity z_0 of a function $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ is called removable if f is holomorphically extendable to all Ω. ie $\exists$ $F: \Omega \to \mathbb{C}$ holom. s.t $F(z) = f(z)$ $\forall z \in \Omega\setminus\{z_0\}$.

We have the following thm of Riemann sometimes called Riemann continuation thm.

Thm Let $z_0 \in \mathbb{C}$. Then the following assertions for a function $f \in \mathcal{H}(\Omega\setminus\{z_0\})$ are equivalent (Ω open, $\neq \emptyset$)

(i) f is holomorphically extendable to Ω

(ii) f is continuously extendable to Ω

(iii) f is bounded in a nbhd of z_0 ie. $\exists$ $r>0$ s.t f is bounded in $D_r^*(z_0)$

(iv) $\lim\limits_{z \to z_0} (z - z_0) f(z) = 0$.

Proof Exercise 5.5 There $\Omega = \mathbb{C}$, $z_0 = 0$ but the proof is verbatim the same. Apply it to $\tilde{f}(z) = f(z) - z_0$.

As a consequence of Riemann's continuation theorem we have

Riemann's thm on removable singularities

Thm 3.1 Suppose that f is holom in an open set Ω except possibly at a point $z_0 \in \Omega$.
If f is bounded in $D_r(z_0) \setminus \{z_0\}$ for some $D_r(z_0) \subset \Omega$ then z_0 is a removable singularity of f.

Ie $\exists$ $F: \Omega \to \mathbb{C}$ holom. s.t $F(z) = f(z)$ $\forall$ $z \in \Omega \setminus \{z_0\}$.

(Note this is just (iii) $\Rightarrow$ (i) of the above thm which says (iii) $\Leftrightarrow$ (i).

Ex ① Let $f(z) = \frac{\sin z}{z}$, $z \neq 0$

Then f has a removable singularity at $z=0$
We can see this either using (iv) ~~def~~ of Riemann's continuation

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \sin z = 0$$

or

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{\cos z}{1} = 1.$$

extention of L'hospital's to $\mathbb{C}$