which imply $f(z) = \frac{\sin z}{z}$ is bdd in a nbhd of 0.

$\lim\limits_{\substack{z \to 0 \\ z \neq 0}} \frac{\sin z}{z} = 1 \Rightarrow$ let $\varepsilon = \frac{1}{2}$ then $\exists\ r > 0$ s.t $\forall\ |z| < r$, $z \neq 0$

$$\left| \frac{\sin z}{z} - 1 \right| < \frac{1}{2},$$

$\Rightarrow \frac{\sin z}{z}$ is bdd in $D_r(0)$

For using $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ $\forall z \in \mathbb{C}$

we get $F(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$ $\forall z$

is the holom. exten. of $\frac{\sin z}{z}$ to all of $\mathbb{C}$.

(2) Another such example of $f(z) = \frac{z}{e^z - 1}$, $z \neq 0$

then f has removable singularity at $z=0$ since $\lim\limits_{z \to 0} f(z) = \lim\limits_{z \to 0} \frac{z}{e^z - 1} = \lim\limits_{z \to 0} \frac{1}{e^z} = 1$

Hence f is bdd in a nbhd of 0,

If f does not have a removable singularity at z_0 and $f \in \mathcal{H}(\Omega \setminus \{z_0\})$

then f is not bounded near z_0.

We can ask whether its unboundedness is similar ~~to~~ to the unboundedness of

$\frac{1}{(z-z_0)^n}$. ie we can ask whether

$(z-z_0)^n f(z)$ is bounded near z_0 for n sufficiently large.

If such an $n \in \mathbb{N}$ exists, then

z_0 is called a pole of f and

the natural number $m := \min\{n \in \mathbb{N} \mid (z-z_0)^n f(z)$ is bdd near $z_0\}$
≥ 1

is called the order of the pole of f at z_0

Poles of first order are called simple poles

eg. $f(z) = (z-z_0)^{-m}$ has a pole of order m at $z = z_0$.

We will see soon that poles arise from reciprocals of holomorphic functions with zeroes

Before we make this more precise let's recall that zeroes of holom. functions are isolated and we have the following theorem for their behaviour near a zero

Thm (1.1 III) Suppose f is holomorphic in a connected open set Ω, and has a zero at a point $z_0 \in \Omega$. And f does not vanish identically on Ω. Then $\exists\ r > 0$ s.t $D_r(z_0) \subset \Omega$ and a unique non-vanishing holom. func $g \in \mathcal{H}(D_r(z_0))$ and a unique positive integer n s.t

$$f(z) = (z - z_0)^n g(z) \qquad \forall z \in D_r(z_0)$$

$$n = \min\{n \mid f^{(n)}(z_0) \neq 0\}$$

The analogous thm for the poles is the following.

Thm (1.2)' For $m \geq \mathbb{N}$, $m \geq 1$ the following statements about $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ are equivalent

(I) f has a pole of order m at z_0 (i.e. $(z-z_0)^m f(z)$ is bounded near z_0 and m is the smallest such integer.

(II) $\exists\, r>0$, $g \in \mathcal{H}(D_r(z_0))$ s.t $g(z_0) \neq 0$

and $f(z) = (z-z_0)^{-m} g(z) \quad \forall z \in D_r^*(z_0)$

(III) $\exists\, r>0$ s.t $D_r(z_0) \subset \Omega$ and $h \in \mathcal{H}(D_r(z_0))$ s.t $h(z) \neq 0$ $\forall z \in D_r^*(z_0)$, h has a zero of order m at z_0 and such that

$$f(z) = \frac{1}{h(z)} \qquad \forall z \in D_r^*(z_0)$$

Proof (I) $\Rightarrow$ (II) f has a pole of order m at z_0 means that $(z-z_0)^m f(z)$ is bounded near z_0, and m is minimal. The thm of Riemann on removable singularity says that $\exists\, g \in \mathcal{H}(D_r(z_0))$ s.t

$$g(z) = (z-z_0)^m f(z) \quad \text{, whenever } z \neq z_0$$

if $g(z_0)$ were zero then it would imply by the previous thm

that $g(z) = (z - z_0)\tilde{g}(z)$ where $\tilde{g}$ is holomorphic in $D_r(z_0)$
Consequently this would give that

$$\tilde{g}(z) = (z - z_0)^{m-1} f(z) \text{ is bounded}$$

near z_0. This would contradict the minimality of m. Hence $g(z_0) \neq 0$

and we get $f(z) = (z - z_0)^{-m} g(z)$

for $z \in D_r^*(z_0)$, and $g(z_0) \neq 0$

(ii) $\Rightarrow$ (iii) Suppose $\exists\, g \in \mathcal{H}(D_s(z_0))$ s.t
$g(z_0) \neq 0$ and
$$f(z) = (z - z_0)^{-m} g(z) \quad \forall z \in D \setminus \{z_0\}$$

Since $g(z_0) \neq 0$, g cont, $\exists\, r > 0$ s.t
$g(z) \neq 0 \quad \forall z \in D_r(z_0)$

Then let $h := \dfrac{(z - z_0)^m}{g(z)} \quad \forall z \in D_r(z_0)$

Then $h(z) \neq 0 \quad \forall z \in D_r^*(z_0)$ and

$h \in \mathcal{H}(D_r(z_0))$

and $\dfrac{1}{h(z)} = g(z)(z-z_0)^{-m} = f(z)$

$\forall z \in D_r^*(z_0)$

Note h has a zero of order m
since $h(z) = (z-z_0)^m (1/g)$
and $\dfrac{1}{g(z)} \neq 0 \quad \forall z \in D_r(z_0)$.

(iii) $\Rightarrow$ (i) Suppose $\exists\, r>0$ s.t

$D_r(z_0) \subset \Omega$ and $h \in \mathcal{H}(D_r(z_0))$
s.t $h(z) \neq 0 \quad \forall z \in D_r^*(z_0)$

$h(z)$ has a zero of order m at z_0
and

$f(z) = \dfrac{1}{h(z)} \quad \forall z \in D_r^*(z_0)$.

Since h has a zero of order m at z_0, $\exists\, g \in \mathcal{H}(D_r(z_0))$ s.t

$h(z) = (z-z_0)^m g(z)$ and $\exists\, s>0$

s.t $g(z) \neq 0 \quad \forall z \in D_s(z_0) \subset D_r(z_0)$

Since g is holom and non vanishing
$1/g$ is holom in $D_s(z_0)$

But then

$$f(z) = \frac{1}{h(z)} = (z - z_0)^{-m} \frac{1}{g(z)} \quad \forall\, z \in D_s^*(z_0)$$

would imply that $(z - z_0)^m f(z) = \frac{1}{g(z)}$

is holom on $D_s^*(z_0)$ and has the holom extension $\frac{1}{g(z)}$ in $D_s(z_0)$

($1/g$ is holom on $D_s(z_0)$ since $g \neq 0$ on $D_s(z_0)$)

By Riemann's extendability thm

$(z - z_0)^m f(z)$ is bounded in a nbhd of z_0.

Moreover $(z - z_0)^{m-1} f(z) = \left(\frac{1}{g(z)}\right)\left(\frac{1}{z - z_0}\right)$

is not bounded since

$\frac{1}{g(z_0)} \neq 0$ and $\frac{1}{z - z_0} \to \infty$ as $z \to z_0$.

Hence m is minimal and f has a pole of order m at z_0

$\boxtimes$

Example ① $f(z) = \dfrac{1}{e^z - 1}$ has a pole of order 1 at $z = 0$

since $\dfrac{1}{f} = e^z - 1 = \displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n!}$

$= z + \dfrac{z^2}{2!} + \dots$

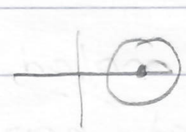
$= z\left(1 + \dfrac{z^2}{2!} + \dots\right)$

has a zero of order 1 at $z = 0$.

(Note $\dfrac{1}{e^z-1}$ has simple poles at $z = 2\pi i n$, $n \in \mathbb{Z}$)

② $f(z) = \dfrac{z}{z^2 - 1}$ has poles of order 1 at $z = \pm 1$

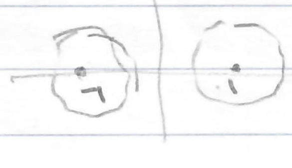
Since $f(z) = (z-1)^{-1}\left(\dfrac{z}{z+1}\right)$



and $h(z) = \dfrac{z}{z+1}$ is holom and non-vanishing $\forall z \in D_{1/2}(1)$

Similarly $f(z) = (z+1)^{-1}\left(\dfrac{z}{z-1}\right)$

and $\tilde{h}(z) = \dfrac{z}{z-1}$ is holom and non vanishing in $D_{1/2}(-1)$



The next theorem is the analog of the power series expansion of a holomorphic function.

Recall if $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$ s.t $D_r(z_0) \subset \Omega$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0) \qquad \forall z \in D_r(z_0)$$

For functions with poles we have

Thm (1.3 III) If f has a pole of order n at z_0 then

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + G(z)$$

whee G is holomorphic in a nbhd of z_0.

Proof: f has a pole of order n at z_0

$\Rightarrow$ we can write $f(z) = (z - z_0)^{-n} g(z)$

$\forall z \in D_r^*(z_0)$ and $g \in \mathcal{H}(D_r(z_0))$ and $g(z_0) \neq 0$

We expand $g(z)$ in a power series