The next theorem is the analog of the power series expansion of a holomorphic function.

Recall if $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$ s.t $D_r(z_0) \subset \Omega$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0) \qquad \forall z \in D_r(z_0)$$

For functions with poles we have

Thm (1.3 III) If $f$ has a pole of order $n$ at $z_0$ then

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + G(z)$$

where $G$ is holomorphic in a nbhd of $z_0$.

Proof: $f$ has a pole of order $n$ at $z_0$

$\Rightarrow$ we can write $f(z) = (z - z_0)^{-n} g(z)$

$\forall z \in D_r^*(z_0)$ and $g \in \mathcal{H}(D_r(z_0))$ and $g(z_0) \neq 0$

We expand $g(z)$ in a power series

$$g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!}(z-z_0)^k \quad , z \in D_r(z_0)$$

Then for $z \in D_r^*(z_0)$

$$f(z) = \frac{1}{(z-z_0)^n}\left[\overbrace{g(z_0)}^{\neq 0} + \frac{g'(z_0)}{1!}(z-z_0) + \cdots + \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots\right]$$

$$= \frac{g(z_0)}{(z-z_0)^n} + \frac{g'(z_0)}{(z-z_0)^{n-1}} + \cdots + \frac{g^{(n-1)}(z_0)/(n-1)!}{(z-z_0)}$$

$$+ \underbrace{\sum_{k=n}^{\infty} \frac{g^{(k)}(z_0)(z-z_0)^{k-n}}{k!}}_{:= G(z)}$$

$$= \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z)$$

$\boxtimes$

Remark: $f(z) = \sum_{k=-n}^{\infty} a_k (z-a)^k$ is a special case of a Laurent series

Defn The number $a_{-1}$ (ie coef. of $(z-z_0)^{-1}$) is called the residue of $f$ at the pole $z_0$, denoted by $\boxed{\operatorname{res}_{z_0} f = a_{-1}}$

The function $\sum_{j=1}^{n} \frac{a_{-j}}{(z-z_0)^j}$ is called

The principal part of $f$ at the pole $z_0$

<u>Remark</u> If $f$ has a pole of order 1 at $z_0$

then $\operatorname{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z)$.

Since if $f$ has a simple pole at $z_0$

then $f(z) = \dfrac{a_{-1}}{z - z_0} + g(z)$, $g \in \mathcal{H}(D_r(z_0))$

Hence $(z - z_0) f(z) = a_{-1} + (z - z_0) g(z)$

$$\lim_{z \to 0} (z - z_0) f(z) = a_{-1} + \lim_{z \to z_0} (z - z_0) g(z)$$

$$= a_{-1}$$

Conversly if $\lim_{z \to z_0} (z - z_0) f(z)$ exists and is non-zero, then

$(z - z_0) f(z)$ is bounded in some nbhd of $z_0$

$z_0$ is a pole of order 1, by our defn of a pole

If the limit exists but is equal to zero then it means $f$ has a removable singularity at $z_0$

More generally we have :

Thm 1.4 If $f$ has a pole of order $n$ at $z_0$

then $\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{(n-1)} \left((z-z_0)^n f(z)\right)$

Proof let $f(z) = P(z) + G(z) \quad z \in D_r(z_0)$

w/ $P(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0}$

and $G(z) \in \mathcal{H}(D_r(z_0))$

Then $(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) +$

$+ \cdots + a_{-1}(z-z_0)^{n-1} + G(z)(z-z_0)^n$

differentiating $(n-1)$ times gives

$$\left(\frac{d}{dz}\right)^{(n-1)} \left((z-z_0)^n f(z)\right) = (n-1)!\, a_{-1} + \underbrace{\frac{d^{(n-1)}}{dz}\left(G(z)(z-z_0)^n\right)}_{\downarrow \text{ product rule.}}$$

$$\lim_{z \to z_0} \left(\frac{d}{dz}\right)^{(n-1)} (z-z_0)^n f(z) = (n-1)!\, a_{-1} + \lim_{z \to z_0} (z-z_0) \tilde{G}(z)$$

hence : $\operatorname{Res}_{z_0} f = a_{-1} = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{(n-1)}}{dz}\left((z-z_0)^n f(z)\right)$

☒

**Example** ① $\text{Res}_i \left(\frac{1}{z^2+1}\right)$

$$= \lim_{z\to i} (z-i)\frac{1}{z^2+1} = \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i}$$

② The function $f = \frac{1}{(z^2+1)^2}$ has double poles at $z = \pm i$

$$\text{Res}_i f = \lim_{z\to i} \frac{d}{dz} \frac{1}{(z+i)^2} = \lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{1}{4i}$$

$$\text{Res}_{-i} f = \lim_{z\to -i} \frac{d}{dz}\left(\frac{1}{(z-i)^2}\right) = \lim_{z\to -i} \frac{-2}{(z-i)^3} = \frac{-2}{(-2i)^3} = \frac{1}{4i^3} = \frac{-1}{4i}$$

**Remark** The following is a useful tool to calculate residues

**Lemma** If $f, g$ are holom at $z_0$, and $g(z)$ has a simple zero at $z_0$ then $\frac{f}{g}$ has a simple pole at $z_0$

and $$\text{Res}_{z_0}\left(\frac{f(z)}{g(z)}\right) = \frac{f(z_0)}{g'(z_0)}$$

Proof. It is clear that if $g$ has simple zero then $g(z) = (z-z_0)\tilde{g}(z)$ where $\tilde{g}(z_0) \neq 0$ and non-zero in some $D_r(z_0)$

and $$\frac{f(z)}{g(z)} = (z-z_0)^{-1} \underbrace{\frac{f(z)}{\tilde{g}(z)}}_{\text{holom in } D_r(z_0)}$$ and non-zero at $z=z_0$

So $\frac{f(z)}{g(z)}$ has a simple pole at $z_0$

We then apply thm 1.4 to $f/g$

$$\mathrm{Res}_{z_0}\left(\frac{f}{g}\right) = \lim_{z \to z_0} (z - z_0)\frac{f(z)}{g(z)}$$

$$\underset{g(z_0)=0}{=} \lim_{z \to z_0} \frac{f(z)(z - z_0)}{g(z) - g(z_0)}$$

$$= f(z_0) \lim_{z \to z_0} \frac{z - z_0}{g(z) - g(z_0)} = \frac{f(z_0)}{g'(z_0)}.$$

$\square$

Example ① $\mathrm{Res}_i\left(\frac{1}{z^2+1}\right) = \mathrm{Res}_i\left(\frac{1}{g(z)}\right) = \frac{1}{g'(i)} = \frac{1}{2i}$

where $g(z) = z^2+1$ has simple zero at $z=i$

② $\mathrm{Res}_i\left(\frac{z^3}{z^2+1}\right) = ?$ We can either use partial fraction expansion

$$\frac{z^3}{z^2+1} = z - \frac{z}{z^2+1} = z - \frac{1}{2}\frac{1}{z-i} - \frac{1}{2}\frac{1}{z+i}$$

and get $\mathrm{Res}_i \frac{z^3}{z^2+1} = -\frac{1}{2}$

or use the above Lemma

$$\mathrm{Res}_i\left(\frac{z^3}{z^2+1}\right) = \frac{f(i)}{g'(i)} = \frac{i^3}{2i} = -1/2$$

with $f = z^3$, $g = z^2+1$

Remark Note if $f(z) = P(z) + G(z)$ $z \in D_r(z_0)$

where $P(z)$ = principal part at pole $z_0$

$G(z)$ - Holomorphic func.

Let $C$ be any circle centered at $z_0$ and contained $D_r(z_0)$

then $$\int_C P(z)\,dz = \int_C \left( \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} \right) dz$$

$$= 2\pi i\, a_{-1}$$

Since $$\int_C \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i & \text{if } n = 1 \end{cases}$$

By Cauchy's thm we also know that if $C \subset D_r(z_0)$ then $\int_C G(z)\,dz = 0$

Hence we have

$$\int_C f\,dz = 2\pi i\, a_{-1}$$

In fact we have the general formula

**Thm (2.1) The Residue Formula** Let $\Omega \subset \mathbb{C}$ open, $F = \{z_0, \dots, z_n\}$ a finite set in $\Omega$.

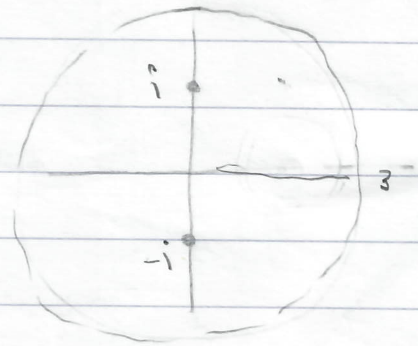
Suppose $f \in \mathcal{H}(\Omega \setminus F)$ holomorphic except for poles at $z_0, z_1, \dots, z_n \in F$. Let $\gamma$ be any circle contained in $\Omega$ with counter clock wise orientation and such that $\gamma \cap F = \phi$. Let $D$ be the open disc bounded by $\gamma$. Then

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{z_i \in F \cap D} \operatorname{res}_{z_i} f}$$

Before we give the proof we'll look at simple examples which uses this formula to calculate integrals.

**Example** 1 Let $\gamma$ be the circle $|z| = 3$

$$\int_\gamma \frac{dz}{(z^2+1)^2}$$




$$= 2\pi i \operatorname{Res}_i \left(\frac{1}{(z^2+1)^2}\right)$$

$$+ 2\pi i \operatorname{Res}_{-i} \left(\frac{1}{(z^2+1)^2}\right)$$

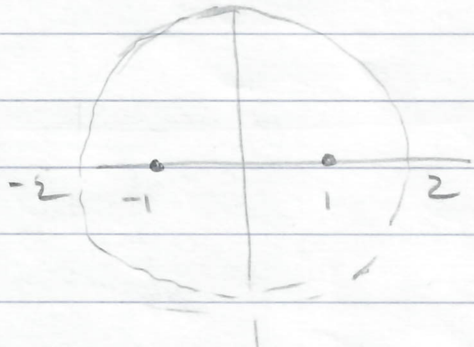
$$= 2\pi i \left[\frac{1}{4i} + \left(\frac{-1}{4i}\right)\right] = 0$$

(2) $$\int_{|z|=3} \frac{z^3}{z^2+1}\,dz = 2\pi i\left(\operatorname{Res}_i \frac{z^3}{z^2+1} + \operatorname{Res}_{-i} \frac{z^3}{z^2+1}\right)$$

$$= 2\pi i\left(\frac{-1}{2} + \frac{-1}{2}\right) = -2\pi i$$

(3) $$\int_{|z-1|=\frac{1}{2}} \frac{dz}{(z^2+1)^2} = 0$$ since there is no pole of $\frac{1}{(z^2+1)^2}$ inside the circle $|z-1|=\frac{1}{2}$

(4) $$\int_{|z|=2} \frac{e^z}{z^2-1}\,dz$$




$$= 2\pi i\left(\operatorname{Res}_1\left(\frac{e^z}{z^2-1}\right) + \operatorname{Res}_{-1}\left(\frac{e^z}{z^2-1}\right)\right)$$

$$\operatorname{Res}_1 \frac{e^z}{z^2-1} = \frac{f(1)}{g'(1)}$$ where $f(z)=e^z$, $g(z)=z^2-1$ using the lemma.

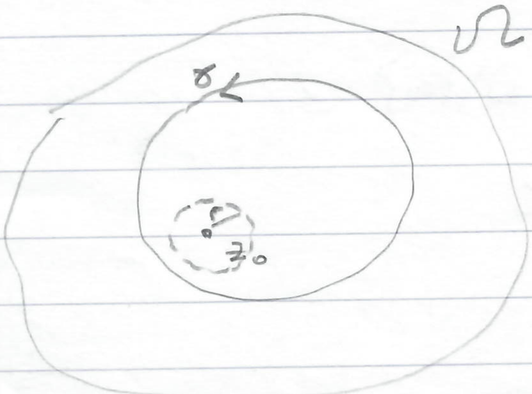
$$= \frac{e}{2}$$

$$\operatorname{Res}_{-1} \frac{e^z}{z^2-1} = \frac{f(-1)}{g'(-1)} = \frac{e^{-1}}{-2}$$

Hence $$\int_{|z|} \frac{e^z}{z^2-1}\,dz = \pi i(e - e^{-1})$$

# Proof of the Residue Formula.

Let's first assume $f$ is holomorphic in an open set $\Omega$ containing a circle and its interior, except for a single pole at $z_0$ inside $\gamma$. Let $D$ the disc bdd by $\gamma$.




By thm 1.3

$$f(z) = P_{z_0}(z) + G(z)$$

where $G(z)$ is holom in a nbhd $D_r(z_0)$ of $z_0$

and $P_{z_0}(z) = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z-z_0}$

is the principal part of $f$ at $z_0$.

Another way to say this is that the function $f(z) - P_{z_0}(z)$ extends holomorphically to $\Omega$.

$$g(z) = \begin{cases} f(z) - P_{z_0}(z) & z \in \Omega \setminus \{z_0\} \\ G(z) & z \in D_r(z_0) \end{cases}$$

is the holom extension of $f(z) - P_{z_0}(z)$ to $\Omega$.

Then $\displaystyle\int_\gamma \left(f(z) - P_{z_0}(z)\right) dz = 0$

$$\int_\gamma f(z)\,dz = \int_\gamma P_{z_0}(z)\,dz$$

and we're left to prove $\int_\gamma P_{z_0}(z)\,dz = 2\pi i\, a_{-1}$

But this follows Cauchy integral formula applied the constant function, $F(z) = 1$

Recall: CIF for derivatives. Let $C = \partial D$ be any circle whose interior $D$ is contained in $\Omega$. Then for $F \in \mathcal{H}(\Omega)$, any $z \in D$

$$F^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{F(w)}{(w - z)^{n+1}}\,dw$$

Hence

$$\int_\gamma \frac{dz}{(z - z_0)^n} = \frac{2\pi i}{n!} \frac{d}{dz}^{(n-1)}(1) = \begin{cases} 0 & \text{if } n-1 \geq 1 \\ 2\pi i & \text{if } n = 1 \end{cases}$$

Hence we get $\int_\gamma f(z)\,dz = 2\pi i\, a_{-1}$.

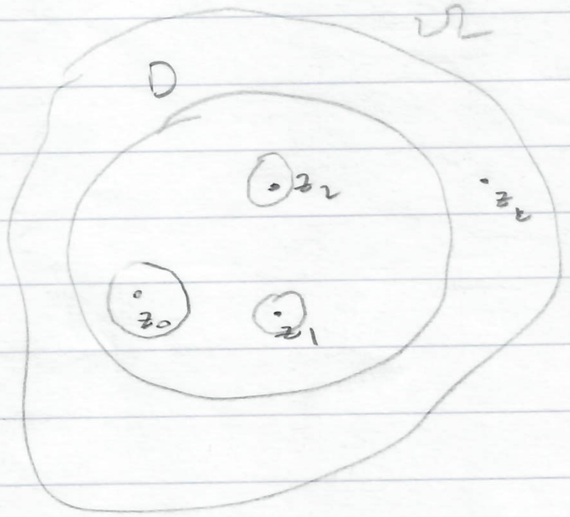
For the general case that $f$ is holom in $\Omega$ except for finitely many points $z_0 \ldots z_n$.

For each $z_i$ let $P_{z_i}$ be the principal part at $z_i$, which is holomorphic in $\mathbb{C} \setminus \{z_i\}$.

Define $g(z) := f(z) - \sum_{z_i \in F} P_{z_i}$ if $z \notin F$.

Then $g \in \mathcal{H}(\Omega \setminus F)$ and in fact $g$ can be extended holomorphically to all $\Omega$.

To see this let $z_0 \in F$, $r > 0$ s.t. $D_r(z_0) \subset D$, $D_r^*(z_0) \cap F = \emptyset$, and $f(z) - P_{z_0}(z)$ is hol. in $D_r(z_0)$




Then for $z \in D_r^*(z_0)$

$$g(z) = \overbrace{\sum_{\substack{z_i \in F \\ z_i \neq z_0}} P_{z_i}(z)}^{\text{hol. in } D_r(z_0)} + \underbrace{f(z) - P_{z_0}(z)}_{\text{hol in } D_r(z_0)}$$

This gives an extension of $g$ to $(\Omega \setminus F) \cup \{z_0\} = \Omega \setminus \{z_1 \ldots z_n\}$

We can do this for each $z_i \in F$ to get a holom ext. of $g$ to all $\Omega$ and by Cauchy's Thm $\int_\gamma g \, dz = 0$

which in return gives

$$\int_\gamma f(z)\,dz = \sum_{z_i \in F} \int_\gamma P_{z_i}(z)\,dz$$

If $z_i \in F \cap D$ then as before

$$\int_\gamma P_{z_i}(z)\,dz = \int_\gamma \sum_{j=1}^{k} \frac{a_{-j}}{(z-z_i)^k}\,dz$$

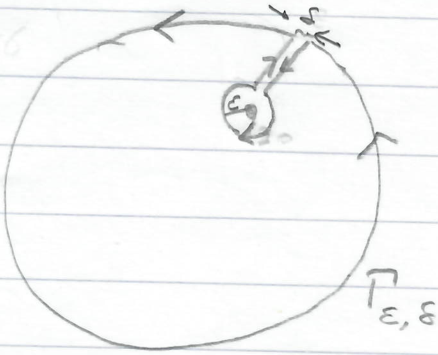
$$= 2\pi i\, a_{-1} = 2\pi i \operatorname{Res}_{z_i} f$$

If $z_i \in F$ but not inside $D$ then

$$\int_\gamma P_{z_i}(z)\,dz = 0$$ since then $P_{z_i}(z)$ is holomorphic inside the disc.

Hence we get $$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_i \in F\cap D} \operatorname{res}_{z_i} f$$

**Remark.** ① Another way to prove this is the following: First assume there is just one pole inside $\gamma$. Consider the following contour $\Gamma_{\varepsilon,\delta}$. Inside $\Gamma_{\varepsilon,\delta}$ $f$ is holom and can show




$$\int_{\Gamma_{\varepsilon,\delta}} f(z)\,dz = 0$$

Here we went around the pole $z_0$ with a small circle of radius $\varepsilon$. The width of the corridor is $\delta$.

We can then make the width of the corridor narrower by letting $\delta \to 0$ and use continuity of $f$ to show that the 2 sides of the corridor cancel each other. The remaining part consists of 2 curves, the large circle $\gamma$ and the small circle $C_\varepsilon$ with cw orientation, and get

$$\int_\gamma f(z) + \int_{C_\varepsilon} f(z)\,dz = 0.$$

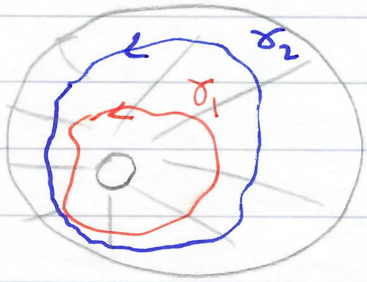
But ~~it~~ takes some effort to make this argument rigorous

Remark The best way to understand and generalize the residue formula (and CIF) is via homotopy -

It is based on the following principle:

Let $f$ be holomorphic in an open set $\Omega$

For example




between 2 circles

Then the principle is that if 2 closed curves can be deformed to each other while remaining in $\Omega$, then $\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$

We'll get back to this soon.

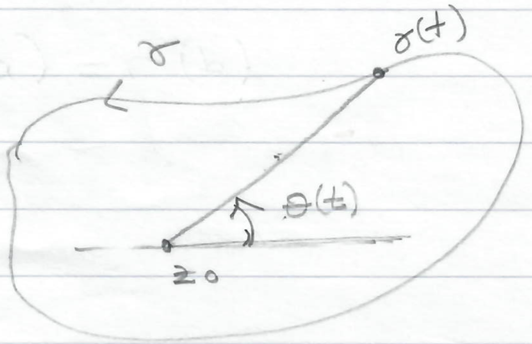
**Remark 3.** If $\gamma$ is not a circle, but a triangle, or a polygon or any curve $\gamma$ which has a parametrization of the form

$$\gamma : [a,b] \longrightarrow \mathbb{C} \setminus \{z_0\}$$

$$t \longmapsto z_0 + r(t)\, e^{i\theta(t)}$$

for some $C^1$ functions $r$, and $\theta : [a,b] \to \mathbb{R}$

$r(t) > 0$, $r(a) = r(b)$, $\theta(a) = 0$, $\theta(b) = 2\pi$




$r(t) := |\gamma(t) - z_0|$

$\theta(t)$ is a continuous choice of argument along $\tilde{\gamma}(t) = \gamma(t) - z_0$.

$$e^{i\theta(t)} = \frac{\gamma(t) - z_0}{|\gamma(t) - z_0|}.$$

Then $\gamma'(t) = r'(t)\, e^{i\theta(t)} + r(t)\, i\, \theta'(t)\, e^{i\theta(t)}$

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = \int_a^b \frac{\gamma'(t)\, dt}{r(t)\, e^{i\theta(t)}}$$

$$= \int_a^b \frac{r'(t)}{r(t)}\, dt + i \int_a^b \theta'(t)\, dt = \log r(t) \Big|_a^b + i\, \theta(t) \Big|_a^b$$

$$= 0 + 2\pi i$$

(This is similar to the parametrization of a circle using a point inside other than the center.)

Note for $\displaystyle\int_\gamma \frac{dz}{(z-z_0)^n} = 0$ for $n>1$

since $\displaystyle\frac{1}{(z-z_0)^{n-1}}\cdot\frac{1}{1-n}$ is a primitive

of $\displaystyle\frac{1}{(z-z_0)^n}$ in $\mathbb{C}\setminus\{z_0\}$, and $\gamma$ is closed.

Hence for any such contour $\gamma$ we have

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_i \in \text{inside } \gamma} \operatorname{res}_{z_i} f$$

—o—

Before we give more theoretical applications of residue theorem, let's give some applications to the evaluation of real integrals.