8-11-23

Example Integrals of rational functions

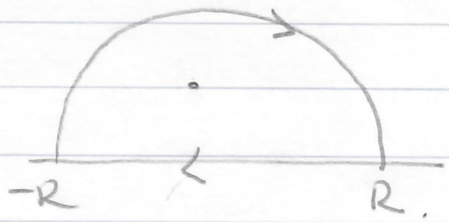
e.g. $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

This is of course using arctangent can be evaluated easily. We give another proof using residue theorem.

Idea: To choose a function f and a closed contour so that part of the contour leads to the real integral after taking limits

In this particular case we take

$f(z) = \dfrac{1}{1+z^2}$ and γ_R as the contour



f has only one pole, at $z = i$ inside γ_R

$$\int_{\gamma_R} f\,dz = 2\pi i \operatorname{Res}_i f = 2\pi i \lim_{z\to i} (z-i)\frac{1}{z^2+1}$$

$$= 2\pi i \lim_{z \to i} \frac{1}{z+i} = \pi.$$

$$\int_{\gamma_R} f(z)\,dz = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{\Gamma_R} \frac{dz}{1+z^2}$$

as $R \to \infty$ the first integral gives $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

and we'll see that over the semicircle Γ_R

the integral goes to zero as $R \to \infty$.

This is because on Γ_R

$$|z^2+1| > R^2 - 1$$

Hence $\frac{1}{z^2+1} < \frac{1}{R^2-1} \sim \frac{1}{R^2}$

$\left| \int_{\Gamma_R} \frac{dz}{1+z^2} \right| < \frac{1}{R^2-1} \cdot \pi R \approx \frac{1}{R}$ and this goes

to zero as $R \to \infty$.

Hence $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$

The same technique works to evaluate the integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

where P, Q polynomials where Q has no zeroes on the real axis.

and $\deg Q(x) \geq \deg P(x) + 2$

Note we need this bound for degrees of P, and Q so that we can get

$$\int_{\Gamma_R} \frac{P(x)}{Q(x)}\,dx \to 0 \quad \text{as } R \to \infty$$

If $\deg Q = n$, $\deg P = m$, on the semicircle for R large

$Q(z)$ satisfy $|Q(z)| > B|z|^n$ for some B

and we can bound $\left|\frac{P(z)}{Q(z)}\right| < C\,\frac{R^m}{R^n}$

$$< C\,\frac{1}{R^{n-m}}$$

Hence $$\left| \int_{\Gamma_R} \frac{P(z)}{Q(z)} \, dz \right| \leq C \frac{1}{R^{n-m}} \cdot R$$

$$= C \frac{1}{R^{n-m-1}}$$

For this to go to zero as $R \to \infty$ we need $n - m - 1 > 0$

Hence $n > m + 1$, ie $n \geq m + 2$

Hence $\deg Q \geq \deg P + 2$.

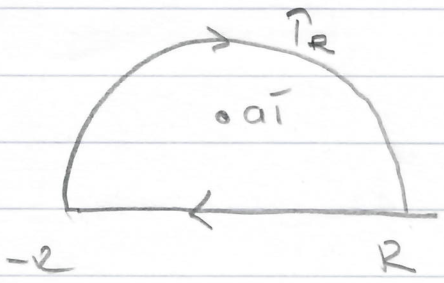
Then $$\int_{\gamma_R} \frac{P(z)}{Q(z)} \, dz = \int_{-R}^{R} \frac{P(x)}{Q(x)} \, dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)} \, dz$$

gives as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx = 2\pi i \sum_{z_i} \operatorname{Res}_{z_i} \frac{P(z)}{Q(z)}$$

where z_i's are the poles of $\frac{P}{Q}$ inside γ_R

Example $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$. $a > 0$



let $f(z) = \dfrac{1}{(z^2+a^2)^2}$

then f has poles at $\pm ai$ of order 2. wlog assume $a>0$.

$$\operatorname*{Res}_{z=ai} \frac{1}{(z^2+a^2)^2} = \lim_{z\to ai} \frac{d}{dz}\left[(z-ai)^2 f(z)\right]$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]$$

$$= \lim_{z\to ai} \frac{-2}{(z+ai)^3} = \frac{-2}{(2ai)^3} = \frac{-2}{-8a^3 i}$$

$$= \frac{-i}{4a^3}.$$

Hence $\displaystyle 2\pi i \operatorname*{Res}_{z=ai} f = \frac{2\pi i(-i)}{4a^3} = \frac{\pi}{2a^3}.$

As above $\displaystyle \lim_{R\to\infty} \int_{\Gamma_R} \frac{dz}{(z^2+a^2)^2} \to 0$ and we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

Example The same contour can be used to evaluate integrals of rational functions times $\sin(ax)$, $\cos(ax)$.

I.e. integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(ax)\, dx .$$

where P, Q are poly in $\mathbb{R}[x]$ with $\deg Q \geq \deg P + 2$.

As $f(z)$ we take $\dfrac{P(z)}{Q(z)} e^{iaz}$

and not $P(z)\cos(az)/Q$ since $\cos az$ behaves badly on the UHP. On the imaginary axis for example

$$\cos it = \frac{e^{t} + e^{-t}}{2} = \frac{e^{2x}+1}{2e^{x}}$$

is hyperbolic cosine which grows exponentially.

Where as $|e^{iz}| = |e^{i(x+iy)}| = e^{-y}$ which is bdd by 1 in the UHP

So $|e^{iz}| \leq 1$ for $\operatorname{Im} z \geq 0$.

e.g. $$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a} \qquad a>0.$$

$$f(z) = \frac{e^{iaz}}{z^2+1}$$ which has only one pole on the UHP, at $z=i$

$$\operatorname*{Res}_{z=i} \frac{e^{iaz}}{z^2+1} = \lim_{z\to i} \frac{e^{iaz}}{z+i} = \frac{e^{-a}}{2i}$$

Hence $$\int_{\gamma_R} f(z)\,dz = 2\pi i \frac{e^{-a}}{2i} = \pi e^{-a}.$$

Since $|e^{iaz}| < 1$ on UHP

$$\left|\frac{e^{iaz}}{z^2+1}\right| \leq \frac{1}{R^2-1}$$ Hence

$$\left|\int_{\Gamma_R} \frac{e^{iaz}}{z^2+1}\,dz\right| \leq \frac{\pi R}{R^2-1} \to 0 \quad \text{as } R\to\infty$$

and $$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a} \qquad a>0$$

Now we note $\dfrac{\cos ax}{1+x^2} = \mathrm{Re}\left(\dfrac{e^{iax}}{x^2+1}\right)$

Hence taking real parts we get

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx = \pi e^{-a}$$

This also shows $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ax}{x^2+1}\,dx = 0$

which can also be seen directly as $\dfrac{\sin ax}{x^2+1}$ is an odd function.

Example Integrals of Trigonometric functions

Residue thm can be used to evaluate real integrals of the form

$$\int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)}\,dt$$

where P, Q are polys and $Q(x,y) \neq 0$ for $x^2+y^2 = 1$ $\forall x, y \in \mathbb{R}$.

Ex. $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} \qquad a>1$

The idea is to convert it to a contour integral around the unit circle

The usual parametrization $z = e^{i\theta}$

gives $dz = ie^{i\theta}d\theta = iz\,d\theta$ and we have

$$d\theta = \frac{dz}{iz}$$

The trig. functions $\cos\theta$, $\sin\theta$ can also be written in terms of z on the unit circle as follows

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-1/z}{2}$$

Hence we can write $a+\cos\theta = a+\frac{1}{2}(z+z^{-1})$

and $$\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} = \int_{|z|=1}\frac{1}{a+\frac{1}{2}(z+z^{-1})}\frac{dz}{iz}$$

$$= \frac{2}{i}\int_{|z|=1}\frac{dz}{z^2+2az+1}$$

(15)

The poles of the integrand are at $-a \pm \sqrt{a^2-1}$. Only one of these roots is inside the unit circle $z_0 = -a + \sqrt{a^2-1}$

$$\mathrm{Res}\left(\frac{1}{z^2+2az+1},\ \underbrace{-a+\sqrt{a^2-1}}_{=z_0}\right) = \lim_{z \to z_0} (z-z_0)\frac{1}{(z-z_0)(z-z_1)}$$

$$= \lim_{z \to z_0} \frac{1}{2z+2a} = \frac{1}{2(-a+\sqrt{a^2-1})+2a}$$

L'hospital

$$= \frac{1}{2\sqrt{a^2-1}}$$

Hence $$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = \frac{2}{i}\,2\pi i\,\frac{1}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}.$$

$\boxtimes$