14-11-23

We now turn to more theoretical applications of residue thm.

We start by giving one more description of an isolated singularity which is a pole.

Namely we have the following

Proposition (Cor 3.2) Suppose f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $\lim_{\substack{z \to z_0 \\ z \neq z_0}} |f(z)| = \infty$

__Proof__ If $f(z)$ has a pole of order $k \geq 1$ at z_0, then

$$f(z) = h(z) \cdot (z - z_0)^{-k} \quad \text{on } D_r^*(z_0)$$

for some $r > 0$ and $h \in \mathcal{H}(D_r(z_0))$ and $h(z_0) \neq 0$

Then $|f(z)| = |h(z)|\, |z - z_0|^{-k} \to \infty$ as $z \to z_0$

Since $|h(z)| \to |h(z_0)| \neq 0$ and $k \geq 1$.

Conversely if $|f(z)| \to \infty$ then we can find $r > 0$ s.t $|f(z)| \geq 1$ for $z \in D_r^*(z_0)$

(in particular $f(z) \neq 0$ for $z \in D_r^*(z_0)$)

so that $h(z) := \dfrac{1}{f(z)}$, $z \in D_r^*(z_0)$

is holom in $D_r^*(z_0)$ and $|h(z)| \leq 1$ there

Furthermore $\lim\limits_{z \to z_0} h(z) = \lim\limits_{z \to z_0} \dfrac{1}{f(z)} = 0$

By Riemann's thm $h(z)$ extends to

a holom func in $D_r(z_0)$ by defining

$h(z_0) = \lim\limits_{z \to z_0} \dfrac{1}{f(z)} = 0$.

If N is the order of zero of h at

z_0, then $f(z) = \dfrac{1}{h(z)}$ has a pole of

order N at z_0. $\boxtimes$.

We've seen an isolated singularity
z_0 of f is removable if

f is bdd near z_0, and z_0 is a pole

if $|f(z)| \to \infty$ as $z \to z_0$.

<u>Defn</u> An isolated singularity which is not
removable or a pole is called
an <u>essential singularity</u>

As we saw in the very beginning the function $e^{1/z}$ near $z=0$ has a more erratic behaviour.

e.g $e^{1/x} \to 0$ as $x \to 0$ from negative reals.

whereas $e^{1/x} \to \infty$ as $x \to 0$ from the +ve reals

In fact any function $f : \Omega \to \mathbb{C}$ behaves erratically near an essential singularity. More precisely we have

Thm (Casorati - Weierstrass) Suppose

f is holomorphic in $D_r^*(z_0)$ and has an essential singularity at z_0.
Then the image of $D_r^*(z_0)$ under f is dense in $\mathbb{C}$.

Remark. The Casorati - Weierstrass thm states that the image of a punctured disc $D_r^*(z_0)$, no matter how small, effectively fills up the whole complex plane. (where z_0 is an essential singularity).
In fact a remarkable thm of Picard says

Thm (Picard) (1879)
If $f \in \mathcal{H}(D_r^*(z_0))$ and has an essential singularity at z_0, then $\mathbb{C} \setminus f(D_r^*(z_0))$ contains at most one point.

The function $f(z) = e^{1/z}$ maps each punctured disc centered at $z=0$ to $\mathbb{C} - \{0\}$
ie it does not take the value 0
so the 'exceptional value' permitted by Picard's thm may in fact exist.

<u>Proof of Casorati - Weierstrass</u> :

w.t.s. $\forall w \in \mathbb{C}$, $\forall \varepsilon > 0$, $\exists z \in D_r^*(z_0)$
s.t $|f(z) - w| < \varepsilon$.

We argue by contradiction and will show that this will force the singularity z_0 to be either removable or a pole and hence contradicting the assumption that z_0 is essential

Assume on the contrary that $\exists w_0 \in \mathbb{C}$ and $\delta > 0$ s.t $\forall z \in D_r^*(z_0)$

$$|f(z) - w_0| \geq \delta$$

let
$$g(z) := \frac{1}{f(z) - w_0} \quad \forall z \in D_r^*(z_0)$$

then on $D_r^*(z_0)$ $g(z)$ is bounded by $1/\delta$
hence has a removable singularity at z_0.
by Riemann's thm on remov. singularities (Thm $\frac{3.1}{\text{III}}$)
Hence there is an extension of g, ie
we can define g at z_0 so that g becomes holom in $D_r(z_0)$

Since $|f(z)-w_0| \geq \delta$ and $g(z) = \frac{1}{f(z)-w_0}$

clearly g is zero free in $D_r^*(z_0)$. Hence

Its reciprocal $1/g$ has an isolated singularity at z_0. This singularity is

either a pole or removable depending on whether $\lim_{z \to z_0} |g(z)| = 0$ or not (resp.)

This in turn gives that the singularity of $f = w_0 + \frac{1}{g}$ at z_0 can be

no worse than a pole, giving the anticipated contradiction ∎ (Note $\lim_{z \to z_0} g(z)$ exists since g has a removable singularity at z_0)

Meromorphic Functions

We now look at functions whose singularities are poles. Since at a pole $\lim_{z \to z_0} |f(z)| = \infty$ this suggests

adding ∞ to the values of functions and and hence include the poles in their domain of definition

eg $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $z \to \frac{1}{z}$ can be extended to $\hat{f} : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$, $z \to \frac{1}{z}$.

Defn (1) $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$

$\underbrace{\{\infty\}}$ "ideal" point at ∞ unsigned.

$\hat{\mathbb{C}}$ is called the extended complex plane

We can supplement the rules in $\mathbb{C}$ by

$\infty \pm z = z \pm \infty = \infty$ for $z \in \mathbb{C}$

$\infty . z = z . \infty = \infty$ for $z \in \hat{\mathbb{C}} \setminus \{0\}$

$z/\infty = 0$ for $z \in \mathbb{C}$

$z/0 = \infty$ for $z \in \hat{\mathbb{C}} \setminus \{0\}$.

The expressions $\infty \pm \infty$, ∞/∞, $0/0$, $0 . \infty$ are not assigned a meaning in $\hat{\mathbb{C}}$.

(2) A sequence $(z_n) \subset \mathbb{C}$ converges to ∞ if

$\lim |z_n| = \infty$ where $(|z_n|)$ is a sequence in $\mathbb{R}$.

Similarly we say $\lim\limits_{z \to z_0} f(z) = \infty$ if

$\lim\limits_{z \to z_0} |f(z)| = \infty$.

<u>Remark</u> $\hat{\mathbb{C}}$ is not a field!

We now define

Defn A function $f : \Omega \to \hat{\mathbb{C}}$, $\Omega \subset \mathbb{C}$ open is called *meromorphic* if the following conditions are satisfied

1. The set $S_f = \{ z \in \Omega \mid f(z) = \infty \} = f^{-1}(\{\infty\})$

   has no limit point in Ω (ie S_f is discrete in Ω)

2. The points in S_f are poles of f

3. The restriction of f to $\Omega \setminus S_f = \{ z \in \Omega \mid f(z) \neq \infty \}$ is holomorphic

   ie $f \in \mathcal{H}(\Omega \setminus S_f)$

Let $\underline{\mathcal{M}(\Omega)}$ = set of all meromorphic functions in Ω.

Example Let $P(z), Q(z) \in \mathbb{C}[z]$ 2 polynomials with no common zeroes.

Note any rational function $\dfrac{p(z)}{q(z)}$ can be reduced to $P(z)/Q(z)$ with no common zeroes.

Let $f(z) = \begin{cases} P(z)/Q(z) & \text{if } Q(z) \neq 0 \\ \infty & \text{if } Q(z) = 0 \end{cases}$

Then $f \in \mathcal{M}(\mathbb{C})$

Since f is holomorphic outside the finite zero set of $Q(z)$

If z_0 is a zero of $Q(z)$, then f has a pole at z_0 since

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{|P(z_0)|}{|Q(z_0)|} = \infty$$

since we assumed $P(z_0) \neq 0$

Hence z_0 is a pole of f.

② The function $f(z) = \cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$

is a meromorphic function in $\mathbb{C}$ with $S_f = \mathbb{Z}$

③ Let $f = \dfrac{e^{1/z}}{z^2 - 1}$. Then f is meromorphic in $\mathbb{C} - \{0\}$, with $S_f = \{\pm 1\}$ but not in $\mathbb{C}$.

If we have 2 functions $f, g \in \mathcal{M}(\Omega)$ with pole sets S_f, S_g

then $(f+g)$ is holomorphic in $\Omega \setminus (S_f \cup S_g)$

to define $f+g$ at pts in $\Omega \setminus (S_f \cup S_g)$ we just use the usual defn

$$(f+g)(z) = f(z) + g(z) \qquad \forall z \in \Omega \setminus (S_f \cup S_g)$$

So we really only have to worry about pts $z \in S_f \cup S_g$

But we can extend $f+g : \Omega \setminus (S_f \cup S_g) \to \mathbb{C}$ to a meromorphic function

$$f+g : \Omega \to \hat{\mathbb{C}} \quad \text{as}$$

follows: if $z_0 \in S_f \cup S_g$, write

$$\left.\begin{aligned} f(z) &= P_f(z) + \tilde{f}(z) \\ g(z) &= P_g(z) + \tilde{g}(z) \end{aligned}\right\} \forall\, z \in D_r^*(z_0)$$

where P_f, P_g are the principal part of f and g at z_0 (one of them can be zero if f, or g does not have a pole at z_0)

$$\tilde{f}, \tilde{g} \in \mathcal{H}(D_r(z_0))$$

Then $$f+g = \underbrace{P_f(z) + P_g(z)}_{\substack{\text{lin. combination} \\ \text{of terms } \frac{1}{(z-z_0)^\ell}}} + \underbrace{\tilde{f}+\tilde{g}}_{\in \mathcal{H}(D_r(z_0))}$$

so $f+g$ has a pole of order ≥ 1 unless $P_f(z) + P_g(z) = 0$ (which can happen)

Hence $f+g \in \mathcal{M}(\Omega)$ with

$$S_{f+g} \subseteq S_f \cup S_g$$

We've proved part (2) of the following proposition.

Proposition $\Omega \subset \mathbb{C}$ open

① $\mathcal{M}(\Omega) \supseteq \mathcal{H}(\Omega)$

② If $f, g \in \mathcal{M}(\Omega)$ then $af + bg \in \mathcal{M}(\Omega)$ for any $a, b \in \mathbb{C}$

Hence $\mathcal{M}(\Omega)$ is a $\mathbb{C}$-vector space.

③ $f, g \in \mathcal{M}(\Omega)$, then $fg \in \mathcal{M}(\Omega)$

④ If $0 \not\equiv f \in \mathcal{M}(\Omega)$ and zeroes of f do not have a limit pt in Ω then $1/f \in \mathcal{M}(\Omega)$.

Proof ① Obvious but note we identified a holom func $f : \Omega \to \mathbb{C}$ with the corresponding function $\tilde{f} : \Omega \to \hat{\mathbb{C}}$ where $\tilde{f} = i \circ f$, $i : \mathbb{C} \hookrightarrow \hat{\mathbb{C}}$.

② The same argument for $f+g$ works with $af+bg$

③ $f = P_f + \tilde{f}$, $g = P_g + \tilde{g}$. Let $z_0 \in S_f \cup S_g$

then $$fg = (P_f + \tilde{f})(P_g + \tilde{g})$$

$$= P_{fg} + G$$ where P_{fg} is a lin comb of $\frac{1}{(z-z_0)^\ell}$

G holom in $D_r(z_0)$.

$$fg = \left(\sum_{k=-n}^{\infty} a_k (z - z_0)^k \right) \left(\sum_{\ell=-m}^{\infty} b_\ell (z - z_0)^\ell \right)$$

$$\sum_{N=-(n+m)}^{\infty} \left(\sum_{\substack{k,\ell \\ k+\ell=N}} a_k b_{N-\ell} \right) (z - z_0)^N$$

e.g. if $f = \dfrac{a_{-1}}{z - z_0} + \sum_{n \geq 0} a_n (z - z_0)^n$

$$g = \frac{b_{-2}}{(z - z_0)^2} + \frac{b_{-1}}{z - z_0} + \sum_{\ell=0}^{\infty} b_\ell (z - z_0)^\ell$$

$$fg = \frac{b_{-2} a_{-1}}{(z - z_0)^3} + \frac{b_{-2} a_0 + a_{-1} b_{-1}}{(z - z_0)^2} + \frac{a_{-1} b_0 + a_0 b_{-1}}{(z - z_0)}$$

$$+ \frac{a_{-1} b_0 + b_{-1} a_0 + b_{-2} a_1}{(z - z_0)} + G$$

where G is holomorphic

Hence fg has a pole of order 3.

Similar to $f+g$, we can define

$$fg = \begin{cases} f(z)g(z) & \text{if } z \in \Omega \setminus (S_f \cup S_g) \\ \infty & \text{if } z \in (S_f \cup S_g) \end{cases}$$

Then fg is meromorphic in $\mathcal{M}(\Omega)$

with $S_{fg} \subseteq S_f \cup S_g$.

(2) If $f \in \mathcal{M}(\Omega)$. If $z_0 \in \Omega \setminus S_f$

and $f(z_0) \neq 0$ then

$\frac{1}{f}$ is holom. at z_0. If $z_0 \in \Omega \setminus S_f$

and $f(z_0) = 0$ then $1/f$ has a pole of order k = order of zero of f at $z_0 \geq 1$.

If $z_0 \in S_f$ then $\left|\frac{1}{f(z)}\right| \xrightarrow[z \to z_0]{} 0$

hence $\frac{1}{f}$ has a removable singularity

at z_0. So if zeroes of f has

no limit point in Ω then the poles

of $\frac{1}{f}$ have no limit point in Ω

and hence $1/f \in \mathcal{M}(\Omega)$.

Rmk. If we assume that $f \not\equiv 0$ in any connected component of Ω, then $1/f \in \mathcal{M}(\Omega)$.

Recall: If $f: \Omega \to \mathbb{C}$, Ω open connected $f \in \mathcal{H}(\Omega)$. Then the zeroes of f do not have a limit point in Ω. (Thm 4.8).

For open connected set Ω

The same is true for $f \in \mathcal{M}(\Omega)$.

Proposition let Ω open connected, $f \in \mathcal{M}(\Omega)$
let $Z = \{ z \in \Omega \mid f(z) = 0 \}$, $f \not\equiv 0$
then Z has no limit point in Ω.

Proof. Assume on the contrary that $\exists$ (z_n) distinct points in Z s.t $\lim z_n = b \in \Omega$.

let S_f = poles of f. Then $f \in \mathcal{H}(\Omega \setminus S_f)$ and $\Omega \setminus S_f$ is open, connected, $f \not\equiv 0$ Hence by the above result we've recalled $b \notin \Omega \setminus S_f$

But now $b \notin S_f$ either since if b is a pole of f then $\lim_{z \to b} |f(z)| = \infty$

means that $|f(z)| > 0$ for $z \in \Omega$ with $|z - b| < \varepsilon$

But this is impossible since if $z_n \to b$ then $|z_n - b| < \varepsilon$ for $n \geq n_0$ and $f(z_n) = 0$.

$\blacksquare$

Remark

Let $f \in \mathcal{M}(\Omega)$, z_0, a pole of f
since S_f has no limit point in Ω
$\exists$ a punctured nbhd $D_r^*(z_0)$ of z_0
s.t

$$D_r^*(z_0) \cap S_f = \phi .$$

If the order of the pole of f at z_0 is k

then $f(z) = (z - z_0)^{-k} g(z)$ with a

analytic function $g(z) \in \mathcal{H}(D_r(z_0))$

Hence locally every meromorphic function is the quotient of 2 holom. functions
Here for $f = \dfrac{g(z)}{(z-z_0)^k}$

It is a non-trivial result that if Ω is open and connected, ie a domain then this is globally possible.

ie For any $f \in \mathcal{M}(\Omega)$, for Ω open connected

$\exists$ $g, h \in \mathcal{H}(\Omega)$ s.t $f = g/h$.

Algebraically we can state this as follows:
Recall if Ω open connected then $\mathcal{H}(\Omega)$ has no zero divisors, it is an integral

domain. It has a quotient field

$$Q(\mathcal{H}(\Omega)) = \left\{ \frac{g}{h} : g, h \in \mathcal{H}(\Omega),\ h \not\equiv 0 \right\}$$

and this quotient field (or field of fractions) is $\mathcal{M}(\Omega)$.

(This is similar to the construction of $\mathbb{Q}$ as field of fractions of the integral domain $\mathbb{Z}$)

Defn Let $\Omega \subset \mathbb{C}$ open, $z_0 \in \Omega$. $f \in \mathcal{M}(\Omega)$, $f \not\equiv 0$. Define the valuation (or order) of f at z_0, denoted by $\text{ord}_{z_0} f$, $\nu_{z_0}(f)$ to be the integer $k \in \mathbb{Z}$ s.t

i) If z_0 is not a pole of f, ie $f(z_0) \neq \infty$ then $k \geq 0$ is the order of vanishing of f at z_0

(ii) If $f(z_0) = \infty$ ie z_0 is a pole then $k \leq -1$ is minus the order of the pole at z_0.

(ie if $\text{ord}_{z_0} f > 0$ then z_0 is a zero
$\text{ord}_{z_0} f < 0$ then z_0 is a pole
$\text{ord}_{z_0} f = 0$ then $f(z_0) \neq 0$, $f(z_0) \neq \infty$.

Combining what we know about (Thm 1.1, and 1.2) the behaviour of functions near zeroes and poles we get

Proposition if $f \in \mathcal{M}(\Omega)$, $f \neq 0$, $z_0 \in \Omega$

1) $\mathrm{ord}_{z_0} k \iff \exists\, r>0$, $h \in \mathcal{H}(D_r(z_0))$ s.t. $h(z_0) \neq 0$ and

$$f(z) = (z-z_0)^k h(z) \quad \forall z \in D_r^*(z_0).$$

($k<0$ if z_0 a pole, $k>0$, if z_0 a zero).

2) $\mathrm{ord}_{z_0}(fg) = \mathrm{ord}_{z_0} f + \mathrm{ord}_{z_0} g$

3) if $f+g \neq 0$ then

$$\mathrm{ord}_{z_0}(f+g) \geq \min\left[\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)\right]$$

Example $f(z) = \dfrac{z}{(e^z-1)^2}$, z has zero of order 1 at 0

$(e^z-1)^2$ has zeroes of order 2 at $z = 2\pi i n$. $n \in \mathbb{Z}$.

$$\mathrm{ord}_0 f = \mathrm{ord}_0 z - \mathrm{ord}_0 (e^z-1)^2$$
$= 1 - 2 = -1$ hence f has a pole of order 1 at $z=0$. For $n \neq 0$

$$\mathrm{ord}_{2\pi i n} f = \mathrm{ord}_{2\pi i n} z - \mathrm{ord}_{2\pi i n} (e^z-1)^2 = 0 - 2 = -2$$

Hence f has pole of order 2 at $2\pi i n$.

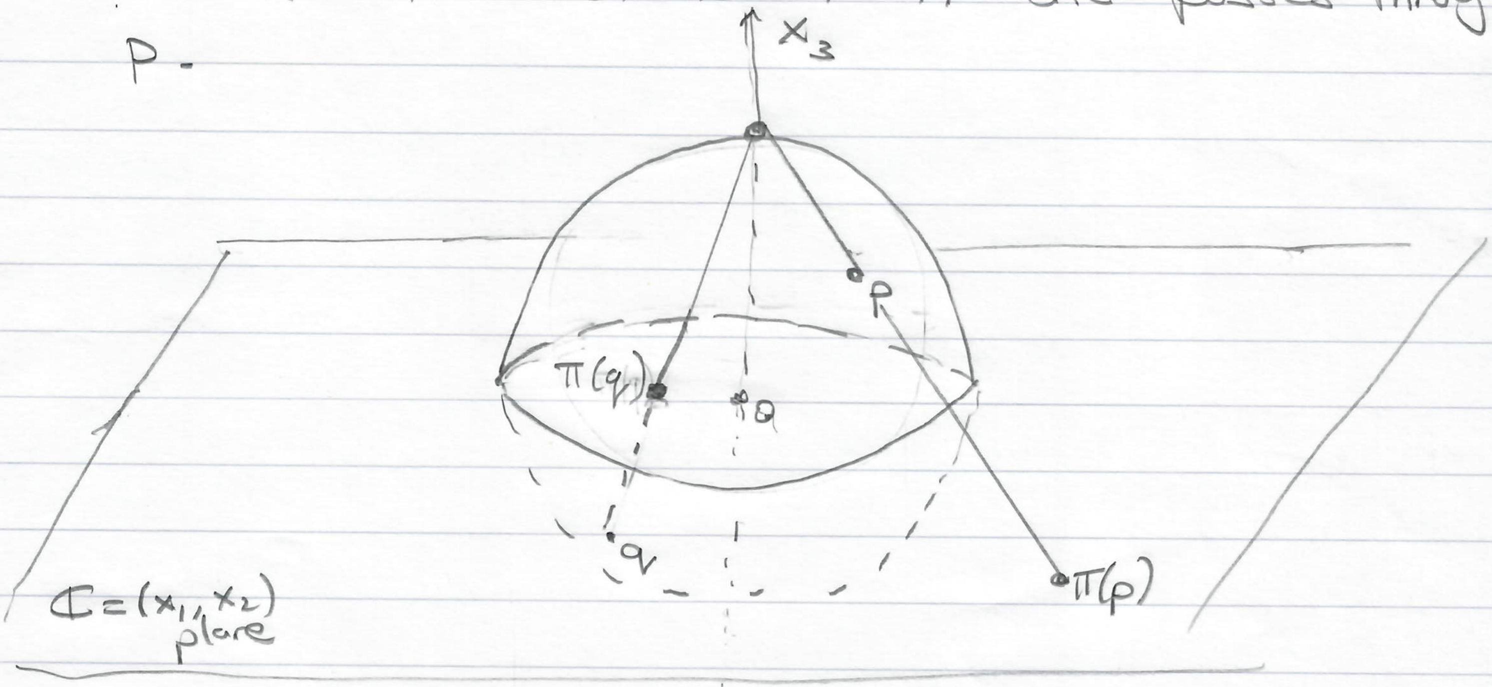
Remark $\hat{\mathbb{C}}$ and the stereographic Projection

Let $S^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$

Identifying $(x_1, x_2, 0)$ with $\mathbb{C}$ we can think of $\mathbb{C}$ sitting in $\mathbb{R}^3$ as the (x_1, x_2)-plane.

Set $N = (0,0,1)$, and define the map

$\pi : S^2 \setminus \{N\} \to \mathbb{C}$ as follows
For $p \in S^2$, $p \neq N = (0,0,1)$, let $\pi(p)$ be the intersection of $\mathbb{C}$ with the ray in $\mathbb{R}^3$ that starts at N and passes through p.



π is called the stereographic projection of $S^2 \setminus \{N\}$ onto $\mathbb{C}$

Explicitly π is given by

$$\pi(p) = \pi(x_1, x_2, x_3) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}, 0\right)$$
$$= \frac{x_1}{1-x_3} + \frac{x_2}{1-x_3} i$$

Note the eqn of the ray that starts at N and go through p is : $N + t(p - N), t \geq 0$
and $\pi(p) = N + t_0(p - N)$ where t_0 is unique positive real number so that
$(0,0,1) + t_0(x_1, x_2, x_3 - 1)$ has 3rd coordinate 0.
Solving for t_0 gives the formula for $\pi(p)$ above.

Defining $\pi(N) = \infty$ gives a bijection

$$\pi : S^2 \to \hat{\mathbb{C}}$$

Conversely given $z \in \mathbb{C}$ one checks that

$$\pi^{-1}(z) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \in S^2 \setminus \{N\}$$

$\pi^{-1}(\infty) := N$ gives the inverse map
Hence we get
S^2 is homeomorphic to $\hat{\mathbb{C}}$.
Since both maps are continuous.