Before we study the values of holomorphic functions using Residue formula let me mention that we can also talk about meromorphic functions on $\hat{\mathbb{C}}$ (as opposed to $\mathcal{M}(\Omega)$, $\Omega \subset \mathbb{C}$)

We have already allowed ∞ as a value of meromorphic functions. We can also allow ∞ in the definition domain and study functions $f : \tilde{\Omega} \longrightarrow \hat{\mathbb{C}}$

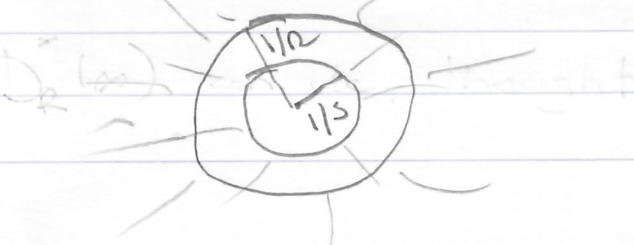
where $\tilde{\Omega} \subset \hat{\mathbb{C}}$.

If a function f is analytic for large values of z, ie $|z| > \frac{1}{R}$ for some $R > 0$, then the function

$$g(z) := f\left(\tfrac{1}{z}\right) \text{ is holomorphic in a}$$

deleted nbhd of 0, $D_R^*(0)$.

(We will denote $\{z \in \mathbb{C} \mid |z| > R^{-1}\}$ by $D_R^*(\infty)$. This notation is designed to have $D_R^*(\infty) \subset D_S^*(\infty)$ when $R < S$.)



Defn For a function f which is analytic for $|z| > 1/R$ for some $R > 0$ we say f has an isolated singularity at ∞ (which will be called removable, a pole or essential if $g(z) := f(\frac{1}{z})$ has an isolated singularity at 0 (which is removable, a pole or essential resp.)

A meromorphic function in the complex plane that is either holomorphic at ∞ or has a pole at ∞ is called meromorphic in $\hat{\mathbb{C}}$

Example ① An entire function is analytic in $D_R^*(\infty)$ for every $R > 0$.

For example $f(z) = e^z$

e^z has an isolated singularity at ∞ which is essential, because

$e^{1/z}$ has an essential singularity at 0.

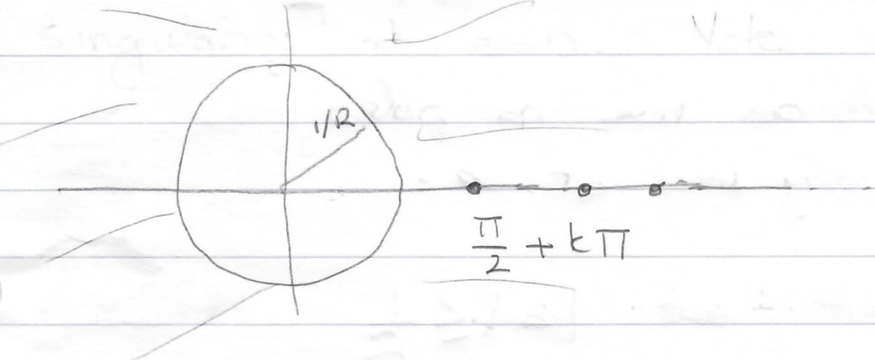
Hence e^z is not meromorphic in $\hat{\mathbb{C}}$.

(It is meromorphic on $\mathbb{C}$)

② $p(z) \in \mathbb{C}[z]$ has a pole at ∞, $p = a_n z^n + \dots + a_0$

$p(\frac{1}{z}) = \frac{a_n}{z^n} + \dots \frac{a_1}{z} + a_0$ has a pole of order n at 0.

(2) $f(z) = \tan z$ does not have an isolated singularity at ∞: Each $D^*_R(\infty)$ includes poles of f; $z = \frac{\pi}{2} + k\pi$



Also note $g(z) = \tan\left(\frac{1}{z}\right)$ have singularities at $S = \left\{\left(\frac{\pi}{2} + k\pi\right)^{-1} \mid k \in \mathbb{Z}\right\}$ which accumulate at $z=0$. The singularity of $\tan\frac{1}{z}$ at $z=0$ is not isolated.

We have the following theorem for meromorphic functions on $\hat{\mathbb{C}}$

Thm 3.4. If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic in the extended complex plane, then it is a rational function.

Clearly each rational function is a meromorphic function on $\hat{\mathbb{C}}$.

Hence $\mathcal{M}(\hat{\mathbb{C}}) = \left\{ \frac{P(z)}{Q(z)} \mid P(z), Q(z) \in \underbrace{\mathbb{C}[z]}_{\text{polynomials}} \right\}$

Proof. Exercise (.

Applications of Residue thm

The first application is called the Argument principle. It uses the residue theorem applied to f'/f, the logarithmic derivative of f to count the zeroes and poles of f inside a curve.

To this end we note the following simple Lemma.

Lemma: Let $\Omega \subset \mathbb{C}$ open and connected $f \in \mathcal{M}(\Omega)$, $f \neq 0$. Then f'/f, the 'logarithmic derivative" of f, is also meromorphic in Ω. And f'/f has poles of order 1 at $z_0 \in \Omega$ for which $\mathrm{ord}_{z_0} f \neq 0$. ie either z_0 is a zero or a pole of f. The residue of f'/f at z_0 is equal to the $\mathrm{ord}_{z_0} f$.

Proof Since $f \neq 0$, Ω open connected zeroes of f do not have a limit point in Ω, and $1/f \in \mathcal{M}(\Omega)$.

Clearly $f' \in \mathcal{M}(\Omega \setminus S_f)$ where S_f is the set of poles of f.

If $z_0 \in S_f$ a pole of order n of f then

$$f(z) = (z - z_0)^{-n} h(z) \qquad \forall z \in D_r^*(z_0)$$

where $h \in \mathcal{H}(D_r(z_0))$ and $h(z_0) \neq 0$.
Then for $z \in D_r^*(z_0)$ we have

$$f'(z) = \frac{-n}{(z - z_0)^{n+1}} h(z) + \frac{h'(z)}{(z - z_0)^n}$$

$$= \underbrace{\left[h'(z)(z - z_0) - n\, h(z)\right]}_{:= \tilde{h}(z)} (z - z_0)^{-(n+1)}$$

$\tilde{h}(z) \in \mathcal{H}(D_r(z_0))$ and $\tilde{h}(z_0) = -n\, h(z_0) \neq 0$

Hence for $\forall z \in D_r^*(z_0)$

$$f'(z) = (z - z_0)^{-(n+1)} \tilde{h}(z)$$

Hence f' has a pole of order $n+1$ at z_0.
(Similarly if f has a zero of order n at z_0 then f' has a zero of order $n-1$ at z_0.)
Hence $f' \in \mathcal{M}(\Omega)$ and so is f'/f

For any $z \in \Omega$, $\operatorname{ord}_{z_0}(f'/f) = \operatorname{ord}_{z_0} f' - \operatorname{ord}_{z_0} f$

$$= \begin{cases} -(n+1) - n = -1 & \text{if } z_0 \text{ is a pole of } f \text{ of order } n \\ (n-1) - n = -1 & \text{if } z_0 \text{ is a zero of } f \text{ order } n \\ \geq 0 & \text{otherwise} \end{cases}$$

Hence f'/f has a pole of order 1 at the points where $\text{ord}_{z_0} f \neq 0$.

We can also calculate the residue using

$$f(z) = (z - z_0)^n g(z) \quad \forall z \in D_r^*(z_0)$$

where $g(z) \in \mathcal{H}(D_r(z_0))$, $g(z) \neq 0$, $\forall z \in D_r(z_0)$
where $n = \text{ord}_{z_0} f$

Hence $n > 0$ if z_0 is a zero
$n < 0$ if z_0 " " pole of f.

$$f'(z) = n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)$$

and
$$\frac{f'}{f} = \frac{n}{(z - z_0)} + \underbrace{\frac{g'(z)}{g(z)}}_{G(z) \in \mathcal{H}(D_r(z_0))} \quad \forall z \in D_r(z_0)$$

Hence
$$\boxed{\text{Res}_{z_0}\left(\frac{f'}{f}\right) = n = \text{ord}_{z_0} f}$$

This lemma immediately gives using the residue thm,

Thm 4.1 (*Argument principle*) Let $\Omega \subset \mathbb{C}$, $\gamma \subset \Omega$, a circle (or any curve s.t. the residue formula holds), $f \in \mathcal{M}(\Omega)$.
If f has no zeroes or poles on γ then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_{z_0 \in Z_f \cap (\text{int}\,\gamma)} \text{ord}_{z_0} f + \sum_{z_0 \in S_f \cap (\text{int}\,\gamma)} \text{ord}_{z_0} f$$

where Z_f = the set of zeroes of f
S_f = " " " poles of f.

Proof. This follows from prev lemma and $\text{Res}_{z_0}\left(\frac{f'}{f}\right) = \text{ord}_{z_0} f$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_{z_i \in \text{int}\,\gamma} \text{Res}_{z_i}\left(\frac{f'}{f}\right)$$

$$= Z - P$$

Z = # of zeroes of f inside γ counted w/ multiplicity

P = # of poles of f inside γ counted w/ multiplicity.