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Cor let $f \in \mathbb{C}[z]$ a polynomial. Choose $R > 0$ large enough so that all zeroes of f are inside $D_R(0)$

Then $$\int_{C_R(0)} \frac{f'}{f}(z) = \deg f$$

We have the following corollary of the Argument principle, which says a holom function when perturbed slightly, it doesn't change its # of zeroes

Thm 4.3 (Rouché's thm)

Suppose f, g are holomorphic in an open set Ω which contains a circle C and its interior. If
$$|f(z)| > |g(z)| \quad \forall z \in C$$

then f, $f+g$ have the same number of zeroes inside C

Proof For $t \in [0,1]$ define
$$f_t(z) = f(z) + tg(z) \quad \text{so that}$$
$f_0(z) = f(z)$, $f_1(z) = f + g$. Note for $z \in C$

Note $$|f_t(z)| = |f(z) + tg(z)|$$
$$\geq \big| |f(z)| - t|g(z)| \big|$$

$$|f_t(z)| \geqslant \big|\, |f(z)| - t\,|g(z)| \,\big| \qquad (|f| > |g| \text{ and } t \leq 1)$$

$$> (1-t)\,|g(z)| \geq 0$$

Hence $|f_t(z)| > 0$ for $z \in C$ hence

$f_t(z)$ has no zeroes on C and

by argument principle $\quad n_t = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f_t'(z)}{f_t(z)}\, dz$

where n_t is the number of zeroes of f_t in C.

Now note n_t is a continuous function of t since $f_t'(z)/f_t(z)$ is jointly continuous for $t \in [0,1]$ and $z \in C$, since both $f_t'(z)$, $f_t(z)$ are jointly continous and $f_t(z) \neq 0$ on C.

(Recall from real analysis:
$f : [a,b] \times [c,d] \to \mathbb{R}$ continous on $[a,b] \times [c,d]$
then
$$h(t) := \int_c^d f(t,x)\, dx \quad \text{is cont. on } [a,b])$$

But n_t is also integer valued. Hence it must be a constant.
(otherwise the intermediate value thm gives the existence of $t_0 \in [0,1]$ s.t n_{t_0} is not integral.)

Hence $$n_0 = \frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = n_1 = \frac{1}{2\pi i}\int_C \frac{(f+g)'}{f+g}\,dz$$

$\blacksquare$

Example Use Rouché's thm to show that the # of zeroes of the polynomial

$p(z) = z^6 + 8z^4 + z^3 + 2z + 3$ inside the unit circle is 4.

We need to express $p(z) = \text{Big} + \text{small}$

on $|z| = 1$. In this case $\text{Big} = 8z^4 = f$

$\text{small} = z^6 + z^3 + 2z + 3 = g$

Note $g(z) = |z^6 + z^3 + 2z + 3| < |8z^4|$

Since for $|z|=1$, $|z^6 + z^3 + 2z + 3| < |z|^6 + |z|^3 + 2|z| + 3$
$= 1 + 1 + 2 + 3$
$= 7 < 8$
$= 8|z|^4$

hence by Rouché thm

$8z^4 = f$ and $f+g = p(z)$ have the same number of zeroes inside the unit circle, since $8z^4$ has 4 zeroes so does p. $\boxtimes$

<u>Example</u> Rouché's thm also gives a simple proof of fund. thm of algebra

let $p(z) = z^d + a_{d-1} z^{d-1} + \dots + a_0$

For $|z|$ large enough the term z^d dominates

Take $f(z) = z^d$, $g(z) = a_{d-1} z^{d-1} + \dots + a_0$

Then $|f(z)| > |g(z)|$ on $|z| = R$

and hence $p(z) = f+g$ and $f = z^d$ has same number of zeroes inside $|z| = R$ since f has d zeroes so does p. $\boxtimes$

The Rouche's thm also leads us to two other important thms.

Thm 4-4 (Open mapping thm) let Ω be open connected, $f \in \mathcal{H}(\Omega)$, f is not a constant, then f is open.

(A map is open if it sends open sets to open sets).

Proof. let $z_0 \in U \subset \Omega$, $f(z_0) = w_0$.

We want to show that a nbhd of w_0 is also contained in $f(U)$.
ie if w is near w_0, w.t.s $\exists\ z \in U$ s.t $w = f(z)$ ie $w \in f(U)$.

Let $r>0$ s.t $\overline{D_r(z_0)} \subset U$ and s.t

$f(z) - w_0 \neq 0 \quad \forall z \in \overline{D^*_r(z_0)}$. This we can do since zeroes of $\tilde{f}(z) := f(z) - w_0$ are isolated.
In particular $f(z) - w_0 \neq 0$ on the circle $C_r(z_0)$.

Since $C_r(z_0)$ is compact, and $f(z) - w_0 \neq 0$ on $C_r(z_0)$, we can find $\delta > 0$
$|f(z) - w_0| \geq \delta$ for z on the circle $C_r(z_0)$ ie if $|z - z_0| = r$.
Now let $w \in \mathbb{C}$ s.t $|w - w_0| < \delta$
ie $w \in D_\delta(w_0)$

let

$$F(z) := f(z) - w = \underbrace{(f(z) - w_0)}_{:= \tilde{f}} + \underbrace{(w_0 - w)}_{\tilde{g}}$$

w.t.s. that $F(z)$ has a zero insite the circle $C_r(z_0)$

This will show that $\exists z \in D_r(z_0)$ s.t $f(z) = w$, Hence $w \in f(D_r(z_0))$

Now we apply Rouche's thm to $\tilde{f}$, $\tilde{g}$ on the circle $|z - z_0| = r$ we have

$$|\tilde{f}| \geq \delta, \quad |\tilde{g}| < \delta$$

Hence on $|z - z_0| = r$, $|\tilde{f}| > |\tilde{g}|$

Since $\tilde{f} = f(z) - w_0$ has a zero inside $D_r(z_0)$ (namely z_0)

$F = \tilde{f} + \tilde{g} = f(z) - w$ also has a zero inside $D_r(z_0)$.

Hence $\exists\ z \in D_r(z_0)$ s.t $w = f(z)$

ie $w \in f(D_r(z_0))$ as wanted $\blacksquare$.

Rmk. This thm for example says that if $f \in \mathcal{H}(D_r(0))$, f not constant then it is not possible that $f(z) \in \mathbb{R}$ for all z since any sset of $\mathbb{R}$ is not open in $\mathbb{C}$.

Thm 4.5 Maximum modulus principle
(Cor 4.6)

Let $\Omega \subset \mathbb{C}$ open connected. $f \in \mathcal{H}(\Omega)$ not constant. Then there is no $z_0 \in \Omega$ s.t

$$|f(z)| \leq f(z_0) \quad \forall z \in \Omega$$

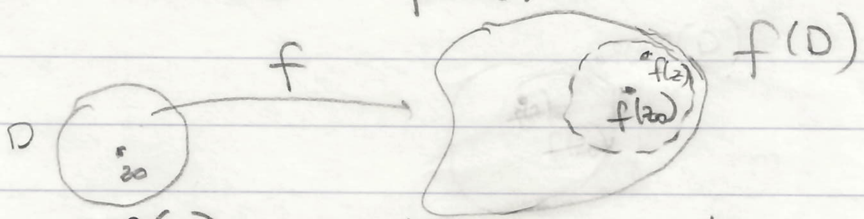
Ie f cannot attain a maximum in Ω.

In particular if $\overline{\Omega}$ is bounded and f is continuous on $\overline{\Omega}$ (holom on Ω) then

$$\underbrace{\max_{z \in \overline{\Omega}} |f(z)|} = \max_{z \in \overline{\Omega} - \Omega} |f(z)|$$

exists because $\overline{\Omega}$ compact

Proof: Suppose $f \in \mathcal{H}(\Omega)$ non constant and Suppose f attained a maximum at $z_0 \in \Omega$. Since by open mapping thm f is an open map, if $D = D_r(z_0) \subset \Omega$, then $f(D)$ is open and contains $f(z_0)$



Hence $f(D)$ contains a disc around $f(z_0)$ but this means there are points $z \in D$ s.t $|f(z_0)| < |f(z)|$

which contradicts that $|f(z)|$ attains its maximum at z_0.

If $\overline{\Omega}$ is bounded and f is nonconstant and continuous on $\overline{\Omega}$ then

$|f(z)|$ attains its maximum on $\overline{\Omega}$ since it is a continuous function on a compact set. By the first part this point where it attains its maximum cannot be inside Ω. Hence it has to be on the boundary $\overline{\Omega} - \Omega$.

If f is constant the statement is trivially true.

<u>Remark</u> The assumption $\overline{\Omega}$ is bdd (ie compact) is crucial

Let $\Omega = \{z \in \mathbb{C} \mid -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}\}$. open connected but $\overline{\Omega}$ is not bounded.

Let $f(z) = \exp(e^z)$, $\left(f|_{\partial\Omega}\right)(z) = \exp(e^{x \pm i\pi/2})$
$= \exp(\pm i e^x)$

Hence $|f(z)| = 1$ for $z \in \partial\Omega$.

but $f(x) = \exp(e^x) \to \infty$ for $x \in \mathbb{R}$ as $x \to \infty$.

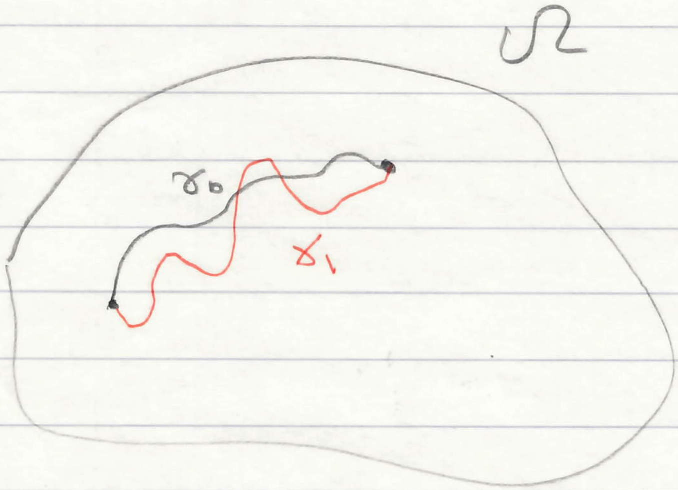
Homotopy and simply connected domains

The key to understand the general form of Cauchy's formula is the idea that if $f : \Omega \to \mathbb{C}$ holomorphic

and if we "continuously deform" γ_0 to γ_1 while staying in Ω and keeping the end points fixed then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

$\gamma_0 : [a, b] \to \Omega$
$\gamma_1 : [a, b] \to \Omega$



Such curves are called homotopic with fixed end points. This means for each $s \in [0, 1]$ $\exists$ a curve $\gamma_s \subset \Omega$

parametrized by $\gamma_s(t)$, $\gamma_s : [a, b] \to \Omega$

s.t. $\gamma_s(a) = \gamma_0(a) = \gamma_1(a)$
$\gamma_s(b) = \gamma_0(b) = \gamma_1(b)$

and at $s = 0$ $\gamma_s(t)\big|_{s=0} = \gamma_0(t)$

and $s = 1$

$\gamma_s(t)\big|_{s=1} = \gamma_1(t)$

This should be done continuously.

Defn (Homotopy) Let $\Omega \subset \mathbb{C}$ open
let $\gamma_0 : [a,b] \to \Omega$, $\gamma_1 : [a,b] \to \Omega$
be 2 curves s.t $\gamma_0(a) = \gamma_1(a)$
$\gamma_0(b) = \gamma_1(b)$

We say γ_0 is homotopic to γ_1 in Ω with endpoints fixed
if $\exists$ $H : [a,b] \times [0,1] \longrightarrow \Omega$
continuous s.t

① $H(t,0) = \gamma_0(t)$ $\forall t \in [a,b]$
$H(t,1) = \gamma_1(t)$

② $H(t,s) := \gamma_s(t)$ is continuous $\forall s \in [0,1]$
$\forall t \in [0,1]$

and $H(a,s) = \gamma_0(a) = \gamma_1(a)$ $\forall s$
$H(b,s) = \gamma_0(b) = \gamma_1(b)$

ie $\gamma_s(t)$ has the same ends points as γ_0, γ_1

Similarly if $\gamma_0 : [a,b] \to \Omega$, $\gamma_1 : [a,b] \to \Omega$
are 2 closed curves, we say
γ_0 is homotopic to γ_1 in Ω if
$\exists$ $H : [a,b] \times [0,1] \to \Omega$ s.t
① $H(t,0) = \gamma_0(t)$ $\forall t \in [a,b]$
$H(t,1) = \gamma_1(t)$

② $\gamma_s(t) := H(t,s)$ is a smooth curve $\gamma : [a,b] \to \Omega$ in Ω and

$$H(a,s) = H(b,s) \quad \forall\, s \in [0,1]$$

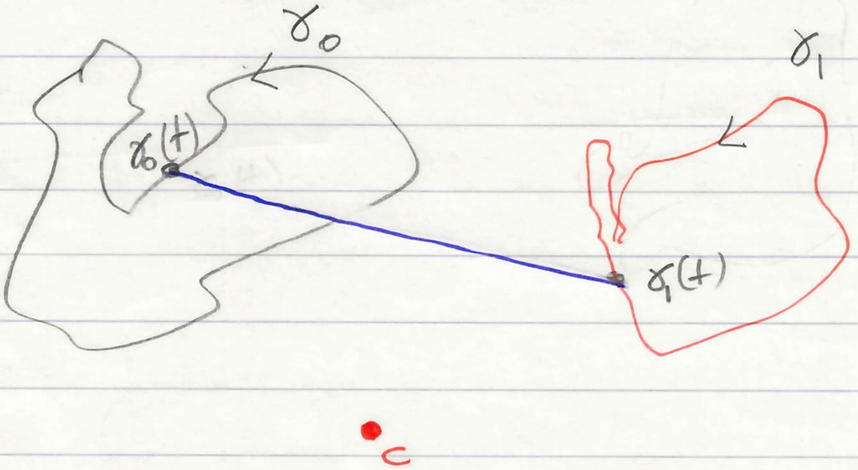
ie

$\gamma_s(t)$ is a closed curve in Ω, $\forall s \in [0,1]$

Examples. ① If $\Omega = \mathbb{C}$ then

any 2 closed curves γ_0, γ_1 are homotopic in particular every closed curve homotopic to the constant curve

$$\sigma : [a,b] \to \mathbb{C} \quad \text{for every } c \in \mathbb{C}.$$
$$t \to c$$



let $H : [a,b] \times [0,1] \longrightarrow \mathbb{C}$

$$(t,s) \longmapsto (1-s)\gamma_0(t) + s\,\gamma_1(t)$$

H is a combination of continuous funcs hence continuous

$$H(t, 0) = \gamma_0(t)$$
$$H(t, 1) = \gamma_1(t)$$

$$H(a, s) = (1-s)\gamma_0(a) + s\,\gamma_1(a)$$
$$H(b, s) = (1-s)\gamma_0(b) + s(\gamma_1(b))$$

Since $\gamma_0(a) = \gamma_0(b)$ and $\gamma_1(a) = \gamma_1(b)$
$H(a, s) = H(b, s)$ $\forall s$
Hence $\gamma_s(t) : [a, b] \to \mathbb{C}$ are all closed curves.

Note geometrically H is defined using the line segment between $\gamma_0(t)$ and $\gamma_1(t)$ for each fixed t. Hence s varies over the line segment between $\gamma_0(t)$ and $\gamma_1(t)$ for fixed t.

For the constant curve σ, we can take the homotopy between σ, γ as

$$H : [a, b] \times [0, 1] \to \mathbb{C}$$
$$(t, s) \longmapsto c(1-s) + s\,\sigma(t)$$

Note the same defn we used for closed curves γ_0, γ_1 also gives a homotopy w/ fixed points if $\gamma_0 : [a, b] \to \Omega$, $\gamma_1 : [a, b] \to \Omega$ are 2 curves w/ fixed end points
ie $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$