§2 Holomorphic Functions

§2.1 Definitions, basic properties

This is a central notion for the rest of the class.

Let $\Omega \subset \mathbb{C}$ an open set
$f : \Omega \to \mathbb{C}$ a complex valued function on Ω.

<u>Defn</u>: f is called <u>holomorphic</u> at $z_0 \in \Omega$ if

$$\lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{f(z) - f(z_0)}{h} = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(z_0+h) - f(z_0)}{h} \text{ exists}$$

Here $h \in \mathbb{C}$, $z_0 + h \in \Omega$, (so that the quotient is well defined)

If the limit exists, we denote it with $f'(z_0)$.
It is called the <u>derivative of f at z_0</u>.

f is called <u>holomorphic on Ω</u> if it is holomorphic $\forall z_0 \in \Omega$.

If f is holomorphic on all of $\mathbb{C}$, then it is called <u>entire</u>.

<u>Remark</u> Regular or complex differentiable are other words used for holomorphic.

Example 2.1 Let $f: \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = z$

Then f is entire. Since

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim \frac{(z_0+h) - z_0}{h} = \lim \frac{h}{h} = 1$$

Hence $f'(z) = 1 \quad \forall z \in \mathbb{C}$.

As in real variables, we have

Prop 2.2. ① Let $\mathcal{H}(\Omega) := \{ f : \Omega \longrightarrow \mathbb{C} \mid f \text{ is hol. on } \Omega \}$

then $\mathcal{H}(\Omega)$ is a $\mathbb{C}$-vector space

More precisely

i.e. if $f, g \in \mathcal{H}(\Omega)$ then $\alpha f + \beta g \in \mathcal{H}(\Omega)$ for every $\alpha, \beta \in \mathbb{C}$.

And

$$(\alpha f + \beta g)' = \alpha f' + \beta g'$$

$(\, \vartheta : \Omega \longrightarrow \mathbb{C}, \; \vartheta(z) = 0$ is the 0-element$)$

② $f, g \in \mathcal{H}(\Omega) \Longrightarrow fg \in \mathcal{H}(\Omega)$

and $(fg)' = f'g + fg'$

③ If $g(z_0) \neq 0$ then f/g is hol. at z_0 and

$$(f/g)' = \frac{f'g - fg'}{g^2}$$

If $g(z) \in \mathcal{H}(\Omega)$ and $g(z) \neq 0 \quad \forall z \in \Omega$

then $f/g \in \mathcal{H}(\Omega)$.

4. If $f : \Omega \to U$, $g : U \to \mathbb{C}$ are holom, Then $g \circ f : \Omega \to \mathbb{C}$ is hol and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z) \quad \forall z \in \Omega.$$

Proofs Very similar to the case of real variables. Here I will just give the proof of 4

Let $z_0 \in \Omega$, $f(z_0) = w_0 \in U$

Consider $F : \Omega \to \mathbb{C}$, $G : U \to \mathbb{C}$ defined by

$$F(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0 \end{cases}$$

$$G(w) = \begin{cases} \dfrac{g(w) - g(w_0)}{w - w_0} & \text{if } w \neq w_0 \\ g'(w_0) & \text{if } w = w_0 \end{cases}$$

Since $\lim\limits_{z \to z_0} F(z) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} = f'(z_0) = F(z_0)$

we have F is continuous at z_0.

Similarly G is cont. at w_0. Hence $G \circ f$ is continoust at z_0. (Since f is diff at z_0, It is continous at z_0.

For $z \in \Omega$, $z \neq z_0$ we have

$$\frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} = \frac{g(f(z)) - g(f(z_0))}{z - z_0}$$

using $w_0 = f(z_0)$

$$= \begin{cases} \dfrac{g(f(z)) - g(w_0)}{f(z) - w_0} \quad \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } f(z) \neq w_0 \\ 0 & \text{if } f(z) = w_0 \end{cases}$$

$$= G(f(z))\, F(z)$$

$\left(\text{Note if } f(z) = w_0 \text{ then } F(z) = \dfrac{w_0 - f(z_0)}{z - z_0} = 0 \right)$

Hence $\displaystyle \lim_{z \to z_0} \frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} = \lim_{z \to z_0} G(f(z)) F(z)$

$$= G(f(z_0)) \cdot F(z_0) \quad \text{(since } G \circ f,\ F \text{ are cont. at } z_0\text{)}$$

$$= G(w_0)\, F(z_0)$$

$$= g'(w_0) \cdot f'(z_0)$$

$$= g'(f(z_0)) \cdot f'(z_0) \quad \text{as wanted}$$

$\square$

Rmk 2.3 Note if $f: \Omega \to \mathbb{C}$
is diff. at $z_0 \in \Omega$ then
$\exists$ a complex number $c \in \mathbb{C}$ s.t

$$f(z) = f(z_0) + c(z - z_0) + E(z; z_0)$$

with $E: \Omega \to \mathbb{C}$ satisfying

$$\lim_{z \to z_0} \left| \frac{E(z, z_0)}{z - z_0} \right| = 0.$$

Here $c = f'(z_0)$

Example 2.4 Example 2.1, Prop 2.2
applied repeatedly show that any
polynomial $p(z) \in \mathbb{C}[x]$ is differentiable

For $p(z) = z^n$ we have

$$p'(z_0) = \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim \frac{(z - z_0)(z^{n-1} + z^{n-2} z_0 + \dots + z_0^{n-1})}{z - z_0}$$

$$= \lim_{z \to z_0} z^{n-1} + z^{n-2} z_0 + \dots + z_0^{n-1} = n z_0^{n-1}$$

② **Important non-example**

Let $f(z) = \bar{z}$. Then

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{\overline{z_0+h} - \overline{z_0}}{h} = \frac{\bar{h}}{h}.$$

For $h = t$, $t \in \mathbb{R}$ this limit is 1
For $h = it$, $t \in \mathbb{R}$ the limit is -1

Hence $\lim \frac{f(z_0+h) - f(z_0)}{h}$ doesn't exist for any $z_0 \in \mathbb{C}$. And $f(z) = \bar{z}$ is not holomorphic.

Note that $f(z) = \bar{z}$ as a function from $\mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$F : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x,y) \mapsto (x,-y).$$

Hence it is a linear function and is differentiable with $J_F(x_0, y_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Recall A function $F : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (u(x,y), v(x,y))$ is differentiable at a point $P_0 = (x_0, y_0)$ if $\exists$ linear map $J : \mathbb{R}^2 \to \mathbb{R}^2$ s.t

$$\lim_{\substack{P \to P_0 \\ P \neq P_0}} \left(\frac{F(P) - F(P_0) - J(P - P_0)}{\|P - P_0\|} \right) = 0.$$

equivalently

$$F(P) - F(P_0) = J(P - P_0) + \psi(P - P_0)|P - P_0|$$

with $|\psi(P - P_0)| \to 0$ as $P \to P_0$

The linear map $J : \mathbb{R}^2 \to \mathbb{R}^2$ is unique and called the differential of F at P_0.
In the standard basis of $\mathbb{R}^2$, J is represented by the Jacobian Matrix of F

$$J = J_P(x, y) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$$

Recall We can view $\mathbb{C}$ as a 1-dim'l vector space over $\mathbb{C}$ with basis $\{1\}$ or as a 2-dimensional real v. space with basis $\{1, i\}$.

And a map $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear if $Tz = \lambda z$ for some $\lambda \in \mathbb{C}$

On the other hand $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear if

$$T(z) = T(x + yi) = T(1)x + T(i)y = \lambda z + \mu \bar{z}$$

with

$$\left.\begin{aligned} \lambda &= \frac{1}{2}\left(T(1) - i\,T(i)\right) \\ \mu &= \frac{1}{2}\left(T(1) + i\,T(i)\right) \end{aligned}\right\} \text{ using } x = \frac{z + \bar{z}}{2},\quad y = \frac{z - \bar{z}}{2i}$$

Hence every $\mathbb{C}$-linear map $T: \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear

But an $\mathbb{R}$-linear map $T: \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear only if $\mu = 0$ ie $T(i) = iT(1)$

If $T(1) = a + bi$ and $T(i) = c + di$
$T(i) = iT(1) \Rightarrow b = -c$ and $a = d$.

If we identify $\mathbb{C}$ with $\mathbb{R}^2$ with $z = x + iy \leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$
Since every $\mathbb{R}$-linear map $\mathbb{R}^2 \to \mathbb{R}^2$ is given by a 2×2 real matrix $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

Such a map is also $\mathbb{C}$-linear if A is of the form $A = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$

Rmk: Note that in Example $f(z) = \bar{z}$ as a map from $\mathbb{R}^2 \to \mathbb{R}^2$, it is differentiable with Jacobian equal to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is not a matrix of the above form!

Our next goal is to see how this linear algebra fact about $\mathbb{R}$-linear versus $\mathbb{C}$-linear maps is reflected in the case of a complex function $f: \mathbb{C} \to \mathbb{C}$ and its differentiability..

§2.2 Cauchy – Riemann Equations

Holomorphicity vs real differentiability

let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic at z_0.

If $f(x+iy) = u + iv$, we can also view f

as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$; $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$(x,y) \mapsto (u(x,y), v(x,y))$$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$

independent of how $z \to z_0$.
In particular we can have z tend to z_0 along the line $z = x + iy_0$ by letting $x \to x_0$

$$f'(z_0) = \lim_{x \to x_0} \frac{f(x+iy_0) - f(x_0+iy_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}$$

$$= u_x(x_0,y_0) + i\, v_x(x_0,y_0)$$

Hence we conclude that the usual partial derivatives $u_x(z_0)$, $v_x(z_0)$ and hence the partial derivative $f_x(z_0) = u_x(x_0) + iv_x(x_0)$ exist

and $\boxed{f'(z_0) = u_x(z_0) + i v_x(z_0) = f_x(z_0)}$ (A)

On the other hand approaching $z_0 = x_0 + iy_0$ along $z = x_0 + iy$ with $y \to y_0$ gives

$$f'(z_0) = \lim_{y \to y_0} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{i(y - y_0)}$$

$$= \lim_{y \to y_0} \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0} - i \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{y - y_0}$$

$$= v_y(z_0) - i u_y(z_0)$$

We obtain that the partial derivatives $f_y(z_0) = u_y(z_0) + i v_y(z_0)$ also exists and

$$\boxed{f'(z_0) = v_y(z_0) - i u_y(z_0) = -i f_y(z_0)} \quad (B)$$

(A) and (B) gives
$$\boxed{\begin{aligned} u_x(z_0) &= v_y(z_0) \\ u_y(z_0) &= -v_x(z_0) \end{aligned}}$$
Cauchy Riemann equations (CR)

If we introduce two differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$