22.11.23

② Note the same formula for the homotopy works for any Ω which is convex.
Ie if we have 2 curves $\gamma_0(t)$, $\gamma_1(t)$ either closed or with fixed end points in a convex set Ω. Then since for a convex set the line segment between any 2 points in also in the set, the function defined by

$$H : [0, b] \times [0, 1] \longrightarrow \Omega$$
$$(t, s) \longmapsto (1-s)\gamma_0(t) + s\gamma_1(t)$$

gives a homotopy in Ω.

In particular this works for Ω a disc.

③ An example of 2 curves which are <u>not</u> homotopic in Ω is if we take $\Omega := \mathbb{C} - \{0\} = \mathbb{C}^*$

$$\gamma_0(t) : [0, \pi] \longrightarrow \Omega$$
$$t \longrightarrow e^{it}$$

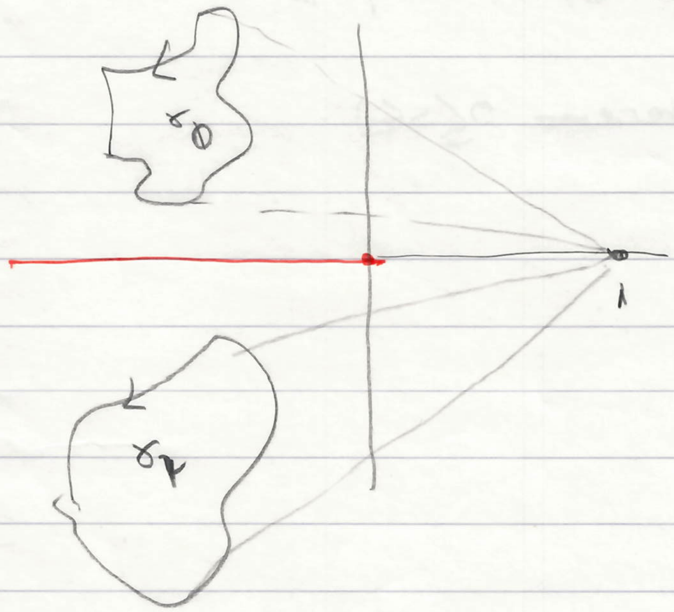
$$\gamma_1(t) : [0, \pi] \longrightarrow \Omega$$
$$t \longrightarrow e^{-it}$$

Then γ_0, γ_1 are not homotopic in Ω

We'll see a simple proof of this when we see the homotopy version of Cauchy's thm.

Intuitively to deform γ_0 to γ_1 we have to go through 0 which is not in Ω.

(4) $\Omega = \mathbb{C} - (-\infty, 0]$. Ω is not convex, so we cannot use the previous formula. But we can still deform γ_0 to γ_1



The idea is to choose any point the real line say 1 and the constant curve

$$\gamma: [a,b] \to \Omega$$
$$t \to 1.$$

we deform γ_0 to 1 then 1 to γ_2

$$H(t,s) := \begin{cases} 1 + (1-2s)(\gamma_0(t) - 1) & 0 \le s \le \frac{1}{2} \\ 1 + (2s-1)(\gamma_1(t) - 1) & \frac{1}{2} < s \le 1 \end{cases}$$

H is continous, the only point to check is $s = \frac{1}{2}$

$$H(t, \tfrac{1}{2}) = 1 = \lim_{s \to \frac{1}{2}} 1 + (2s-1)(\gamma_1(t) - 1) = 1$$

To see the image of $H(t,s)$ is contained in Ω $\forall$ $a \le t \le b$, $0 \le s \le 1$ check

for example that if $a \le t \le b$, $0 \le s \le \frac{1}{2}$

then if $H(t,s) \notin \Omega$ mean for some t, s

$H(t,s)$ is a real number which is non-positive

ie $1 + (1-2s)(\gamma_0(t) - 1) \le 0$

$\iff (1-2s)(\gamma_0(t) - 1) \le -1$

$\iff \gamma_0(t) \le \dfrac{-1}{1-2s} + 1 = 1 + \dfrac{1}{2s-1} = \dfrac{2s}{2s-1} \le -1$

but $0 \le s \le \frac{1}{2} \Rightarrow 2s \ge 0$, $2s - 1 \le 0$

Hence $\gamma_0(t) \le 0$ but $\gamma_0(t) \in \Omega$ so this cannot happen

$\frac{1}{2} < s \le 1$ is similar

<u>Remark</u> - If γ_0 is homotopic to γ_1 in Ω (either closed or w/ fixed end points) we write $\gamma_0 \sim_{\Omega} \gamma_1$

and simply write $\gamma_0 \sim \gamma_1$ if Ω is fixed and clear.

Then "$\sim$" is an equivalence relation

If $\gamma_0 \sim \gamma_1$ with $H(t, s)$ then

$\gamma_1 \sim \gamma_0$ with $\tilde{H}(t, s) = H(t, 1-s)$

If $\gamma_0 \sim \gamma_1$ with F, $\gamma_1 \sim \gamma_2$ with G

define
$$H(t,s) := \begin{cases} F(t, 2s) & 0 \le s \le \frac{1}{2} \\ G(t, 2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

then H gives a homotopy between γ_0 and γ_2.

The Homotopy thm

We can now state the homotopy thm.

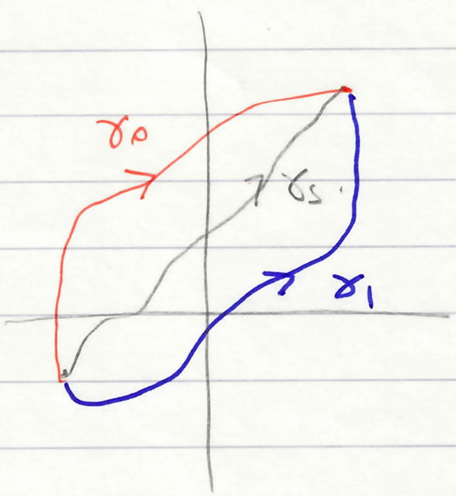
Thm 5_1 (Chap 3) let $\Omega \subset \mathbb{C}$ open
γ_0, γ_1 two curves $\gamma_i : [0, b] \to \Omega$
s.t ① either γ_0, γ_1 are closed and homotopic

or

② γ_0, γ_1 have the same end points and are homotopic with fixed end points.

Then for $f \in \mathcal{H}(\Omega)$ we that

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_2} f(z)\,dz .$$



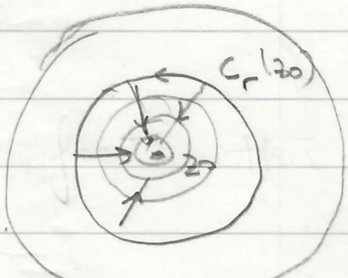
Example ① let $\Omega = D_R(z_0)$, $\varepsilon > 0$
let

$$\gamma_0 = C_r(z_0), \quad 0 < r < R. \text{ Then } \gamma_0$$

can be deformed into the point z_0 (by dilation which can be thought as the constant curve $\gamma_1(t) = z_0$ $\forall t \in [a,b]$.

So $$\int_{C_r(z_0)} f\,dz = \int_{C_0(z_0)} f\,dz$$

$= 0$ since $\gamma_1'(t) = 0$ $\forall t$.



$H(t,s) : [0, 2\pi] \times [0,1] \to D$
$(t,s) \to (1-s)e^{it} + s z_0$

In fact in D any closed curve γ is homotopic to a constant curve, Hence $\int_\gamma f = 0$.

② let $\Omega = \mathbb{C} - \{0\}$

$\gamma_0(t) : [0, \pi] \to \Omega$, $t \to e^{it}$ $\qquad \gamma_1 : [0, \pi] \to \Omega$, $t \to e^{-it}$.

They are <u>NOT</u> homotopic (with fixed end points since if they <u>were</u> then we would get that for $\frac{1}{z} \in \mathcal{H}(\Omega)$

$$\int_{\gamma_0} \frac{1}{z}\,dz = \int_{\gamma_1} \frac{1}{z}\,dz$$

$$\Rightarrow \int_{\gamma_0} f\,dz - \int_{\gamma_1} f\,dz = 0$$

$$\int_{C_1(0)} f(z)dz \quad . \text{ But } \quad \int_{C_1(0)} f(z)\,dz = 2\pi i \neq 0.$$

We now look at the proof of the homotopy thm. We'll look at the case of closed curves. (The book does fixed end points)

Proof of homotopy thm: ① Simpler version:

If we also assume that the homotopy $H(t,s)$ also has continuous 2nd partial derivatives and hence

$$\frac{\partial^2 H}{\partial t\,\partial s} = \frac{\partial^2 H}{\partial s\,\partial t} \quad \forall (t,s) \in [a,b]\times[0,1]$$

then we can give a simpler proof.

For this first recall from real analysis

$h : [a,b]\times[0,1] \to \mathbb{R}$ be a function

suppose $\frac{\partial h}{\partial s}$ exists and continuous

in $[a,b]\times[0,1]$. If we define

$G : [0,1] \longrightarrow \mathbb{R}$

$s \longrightarrow G(s) := \int_a^b h(t,s)\,dt$

then G is differentiable and

$$G'(s) = \int_a^b \frac{\partial h}{\partial s}(t,s)\,dt$$

Applying this to real and imaginary parts we define

$$I(s) := \int_a^b \overbrace{f(H(t,s)).\frac{\partial H}{\partial t}(t,s)}^{h(t,s)}\,dt$$

$$= \int_a^b f(\gamma_s)\,\gamma_s'(t)\,dt = \int_{\gamma_s} f(z)\,dz$$

Note $I(0) = \int_{\gamma_0} f(z)\,dz$ $\quad I(1) = \int_{\gamma_1} f(z)\,dz$

We want to show $I(s)$ is constant.

$$I'(s) = \int_a^b \frac{\partial}{\partial s}\left(f(H(t,s))\frac{\partial H}{\partial t}(t,s)\right)dt$$

$$= \int_a^b \left(\frac{\partial}{\partial s}\left[(f\circ H)\frac{\partial H}{\partial t}\right]\right)dt$$

$$I'(s) = \int_a^b \left[f'(H(t,s)) \frac{\partial H}{\partial s}(t,s) \frac{\partial H}{\partial t}(t,s) + \right.$$

$$\left. f(H(t,s)) \frac{\partial^2 H}{\partial s \, \partial t}(t,s) \right] dt$$

But note what is inside the parenthesis $[\cdots]$ is also equal to

$$\frac{\partial}{\partial t}\left[f(H(t,s)) \frac{\partial H}{\partial s}(t,s) \right]$$

Hence $\displaystyle I'(s) = \int_a^b \frac{\partial}{\partial t}\left[f(H(t,s)) \frac{\partial H}{\partial s}(t,s) \right]$

$$= f(H(t,s)) \frac{\partial H}{\partial s}(t,s) \Big|_{t=a}^{t=b}$$

$$= f(H(b,s) \frac{\partial H}{\partial s}(b,s) -$$

$$f(H(a,s)) \frac{\partial H}{\partial s}(a,s) = 0.$$

Since $H(t,s)$ is homotopy of closed curves $\gamma_s(a) = H(a,s) = H(b,s) = \gamma_s(b)$ for every $s \in [0,1]$, we also have that

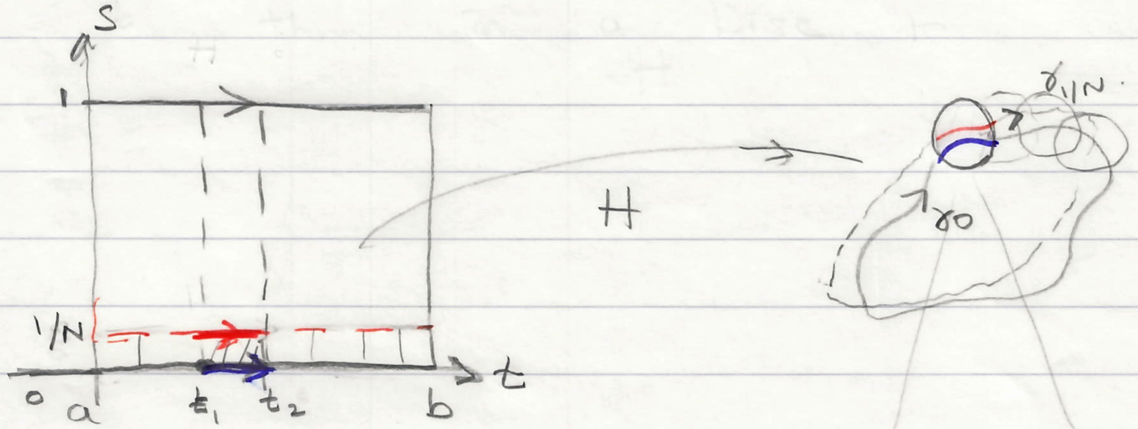
$$\frac{\partial H(a,s)}{\partial s} = \frac{\partial H}{\partial s}(b,s) \quad \forall s$$

Hence $I'(s) = 0 \quad \forall s \in [0,1]$
and $I(s)$ is constant, in particular

$$\int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz$$

☒.

For the general proof the idea is the following: If we make a small deformation of one of the curves $\gamma_s(t)$, say $\gamma_0(t)$ to $\gamma_{1/N}(t)$ so that if we look at a small piece around a point of $\gamma_0(t)$ say $t_1 < t < t_2$ then we can show these are contained in a small disc in Ω.

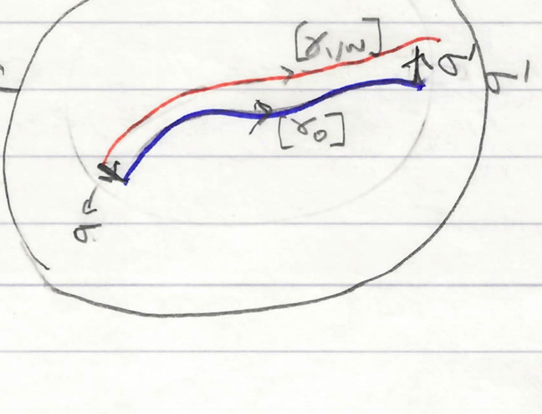


$$H(t, 0) = \gamma_0(t)$$

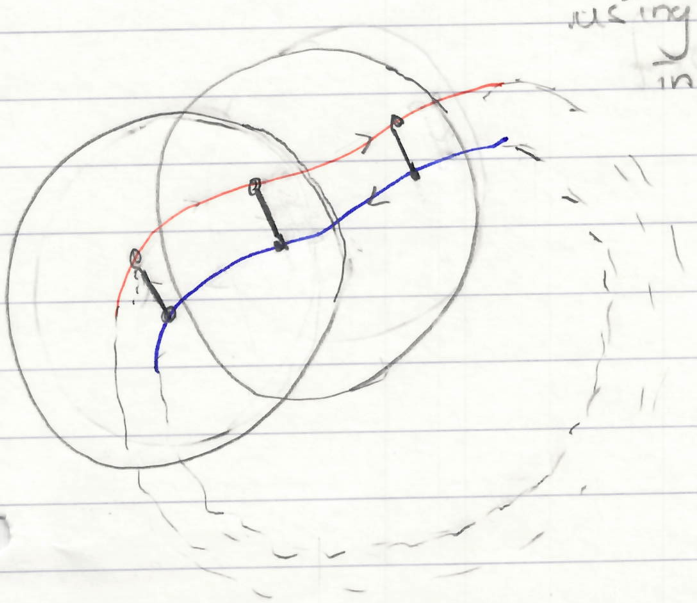
$$H(t, 1) = \gamma_1(t)$$

We can apply Cauchy's thm in a disc to get

$$\int_{[\gamma_0]} f(z) dz = \int_{[\gamma_{1/N}] + \sigma + \sigma'} f(z) dz$$



Now we move over the whole curve $\gamma_0, \gamma_{1/N}$ using small discs contained in Ω to get



$$\int_{\gamma_0} f\, dz = \int_{\gamma_{1/N}} f\, dz.$$

Now we make this idea more precise:

For this we use 2 facts

① If $K = \text{Image } H = H([a,b]\times[0,1])$ then K is compact (Since H is continous and $[a,b]\times[0,1]$ is compact)

This gives the following

Lemma $\exists\ \varepsilon > 0$ s.t $\forall z \in K$ the disc $D_\varepsilon(z)$ is contained in Ω.

Proof: Assume on the contrary no such ε exists. Then $\forall n \geq 1$, $\exists\ z_n \in K$ s.t $D_{1/n}(z_n)$ is not contained in Ω

I.e. $\exists\ w_n \in \mathbb{C} - \Omega$ such that $|z_n - w_n| \leq \frac{1}{n}$