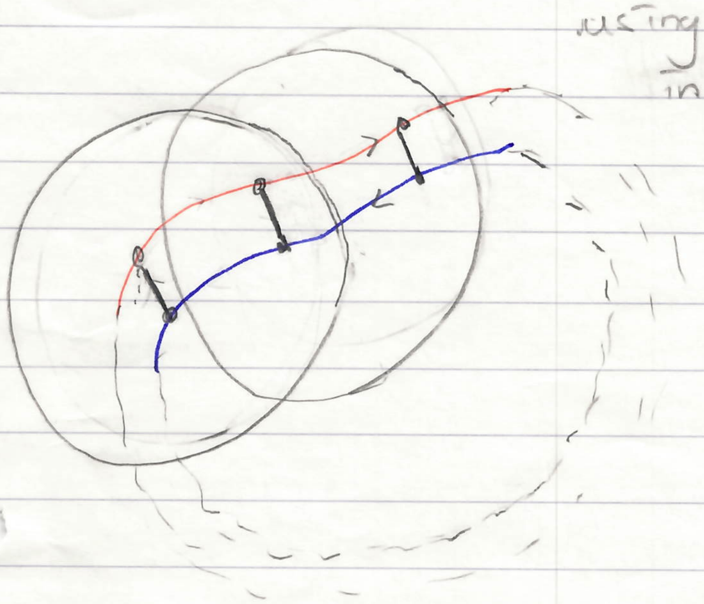
Now we move over the whole curve $\gamma_0, \gamma_{1/N}$ using small discs contained in Ω to get



$$\int_{\gamma_0} f \, dz = \int_{\gamma_{1/N}} f \, dz.$$

Now we make this idea more precise:

For this we use 2 facts

① If $K = \text{Image } H = H([a,b] \times [0,1])$ then K is compact (sine H is continous and $[a,b] \times [0,1]$ is compact)

This gives the following

Lemma $\exists \; \varepsilon > 0$ s.t $\forall z \in K$ the disc $D_\varepsilon(z)$ is contained in Ω.

Proof: Assume on the contrary no such ε exists. Then $\forall n \geq 1$, $\exists \; z_n \in K$ s.t $D_{1/n}(z_n)$ is not contained in Ω

i.e $\exists \; w_n \in \mathbb{C} - \Omega$ such that $|z_n - w_n| \leq \frac{1}{n}$

(z_n) is a sequence in K, K is compact

Hence $\exists$ a conv. subseq, $\left(z_{n_k}\right)_{k \geq 1}$ s.t

$\lim z_{n_k} = z$. Since K is closed, $z \in K$.

But now, since $|w_n - z_n| \leq \frac{1}{n}$

$|w_{n_k} - z_{n_k}| \leq \frac{1}{n}$ Hence $w_{n_k} \to z$ as well.

$w_{n_k} \in \mathbb{C} - \Omega$ which is also closed since Ω is open.

Hence $z \in \mathbb{C} - \Omega$. But this is a contradiction since $z \in K \subset \Omega$.

$\blacksquare$.

(2) This lemma, together with the fact that H is uniformly continuous on the compact set $[a, b] \times [0, 1]$ will now allow us to find the small discs that are contained in Ω.

This is because we can divide the rectangle $[a, b] \times [0, 1]$ into small rectangles s.t the image of these small rectangles are contained in small discs of radius ε.

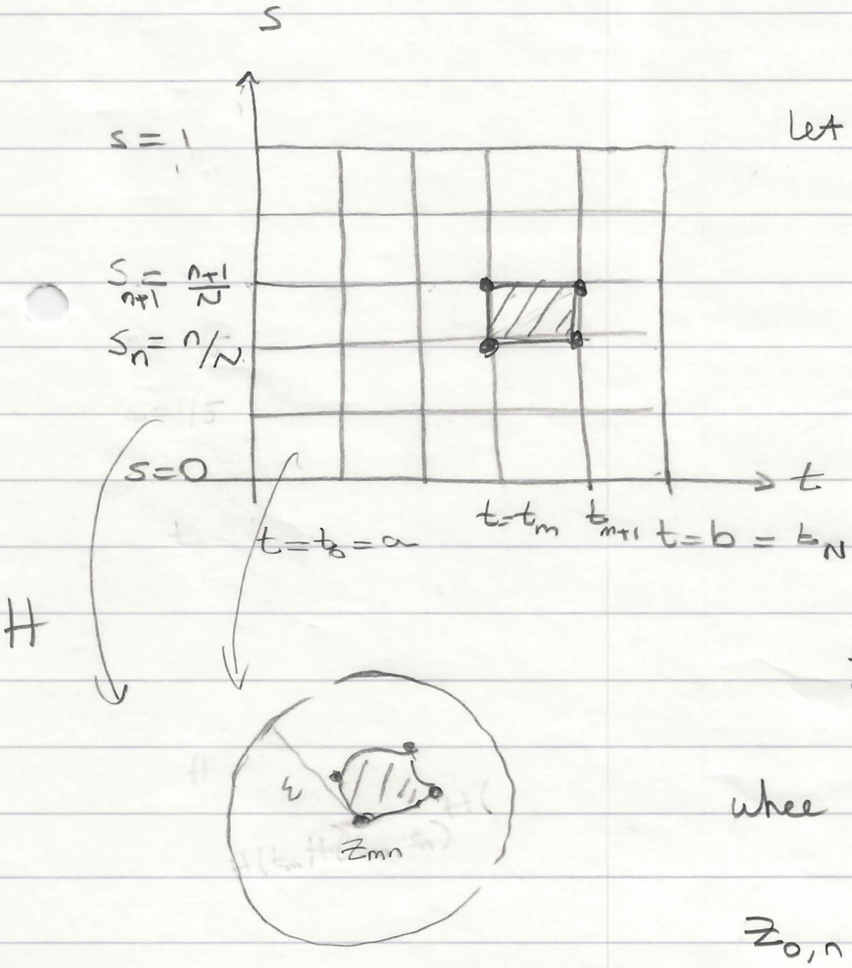
More precisely, since H is unif cont on $[a,b] \times [0,1]$, $\exists\, N > 0$ s.t

$$\Big| H(t,s) - \overbrace{H(t_m, s_n)}^{:= z_{mn}} \Big| < \varepsilon \qquad (*)$$

whenever $\quad |(t,s) - (t_m, s_n)| < \dfrac{2}{N}$

where $\quad t_m = a + \dfrac{b-a}{N} \cdot m \qquad 0 \le m \le N$

$$s_n = \frac{n}{N} \quad , \quad 0 \le n \le N$$



Let $\quad Q_{mn} = [t_m, t_{m+1}] \times [s_n, s_{n+1}]$

Since the diameter of Q_{mn} is $\dfrac{\sqrt{2}}{N}$.

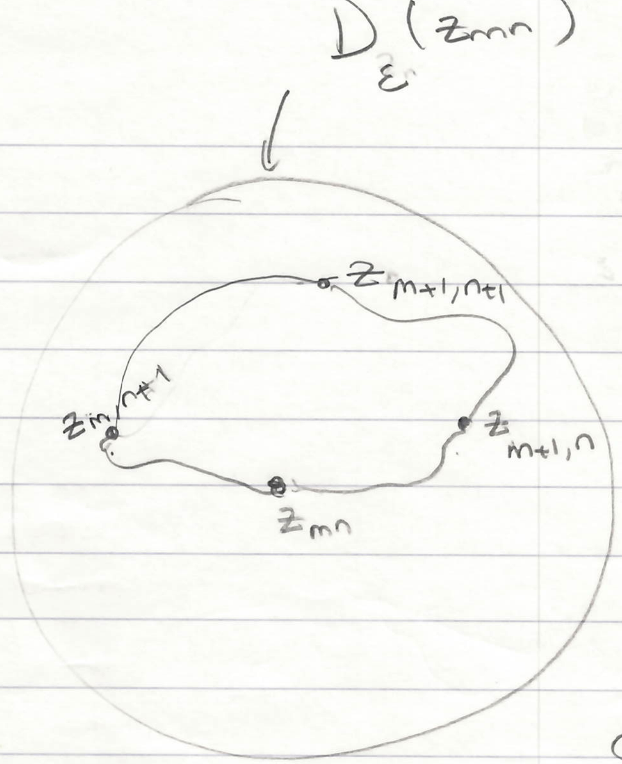
it follows that from $(*)$ that

$$H(Q_{mn}) \subset D_{\varepsilon}(z_{mn})$$

where $\quad z_{mn} := H(\underbrace{t_m, s_n}_{:= x_{mn}})$

$z_{0,n} = H[a, n/N]$, i.e. $z_{0,n} = z_{N,n}$

$z_{N,n} = H[b, n/N]$



We use now induction on n, $0 \le n \le N$ to show that

$$\int_{\gamma_{n/N}} f(z)\,dz = \int_{\gamma_0} f(z)\,dz$$

Clearly true for $n = 0$.

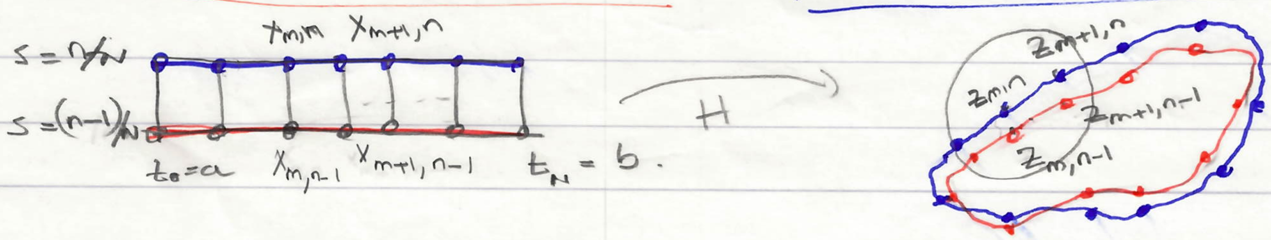
Assume $n \ge 1$ and it holds that

$$\int_{\gamma_{\frac{n-1}{N}}} f(z)\,dz = \int_{\gamma_0} f(z)\,dz$$

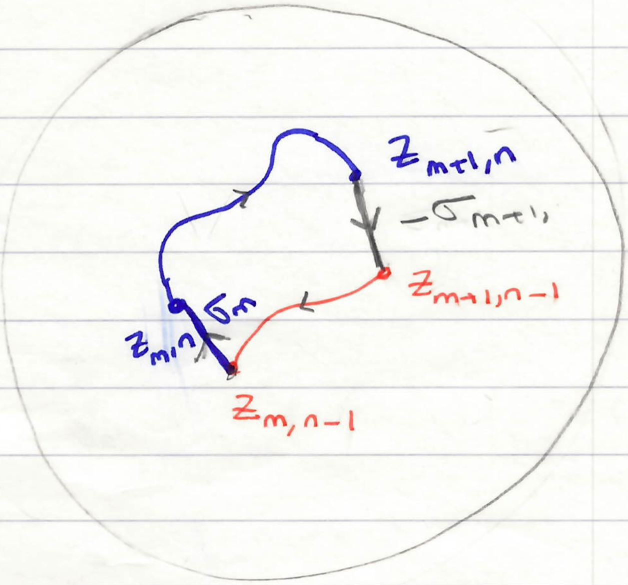
It is then enough to show that

$$\int_{\gamma_{\frac{n-1}{N}}} f(z)\,dz = \int_{\gamma_{n/N}} f(z)\,dz$$

$$\boxed{\gamma_{\frac{n-1}{N}} = H\left(t, \frac{n-1}{N}\right)} \qquad \boxed{\gamma_{n/N} = H\left(t, \frac{n}{N}\right)}$$



For each $0 \le m \le N$,



let $\boxed{\gamma^{(m)}_{\frac{n-1}{N}} = \gamma_{\frac{n-1}{N}}\Big|_{[t_m, t_{m+1}]}}$

and $\boxed{\gamma^{(m)}_{\frac{n}{N}} = \gamma_{\frac{n}{N}}\Big|_{[t_m, t_{m+1}]}}$

Let σ_m = line segment between $z_{m,n-1}$ and $z_{m,n}$

σ_{m+1} = line seg. between $z_{m+1,n-1}$, $z_{m+1,n}$

$\sigma_0 = [z_{0,n-1}, z_{0,n}]$

$\sigma_N = [z_{N,n-1}, z_{N,n}]$

Cauchy's thm on the disc $D_\varepsilon(z_{m,n-1})$ gives

$$\int_{\gamma^{(m)}_{\frac{n}{N}}} f - \int_{\sigma_{m+1}} f - \int_{\gamma^{(m)}_{\frac{n-1}{N}}} f + \int_{\sigma_m} f = 0$$

$$\int_{\gamma_{\frac{n-1}{N}}} f = \sum_{m=0}^{N-1} \int_{\gamma^{(m)}_{\frac{n-1}{N}}} f\,dz = \sum_{m=0}^{N-1} \int_{\gamma^{(m)}_{\frac{n}{N}}} f\,dz$$

$$+ \sum_{m=0}^{N-1} \left(\int_{\sigma_m} f - \int_{\sigma_{m+1}} f \right)$$

$$= \int_{\gamma_{\frac{n}{N}}} f + \int_{\sigma_0} f - \int_{\sigma_N} f = \int_{\gamma_{\frac{n}{N}}} f \, dz$$

$$H(a, \tfrac{n-1}{N}) = H(b, (n-1)/N)$$

Now $\sigma_0 = \sigma_N$ since $\gamma_{\frac{n-1}{N}}(a) = \gamma_{\frac{n-1}{N}}(b)$

and similarly $\gamma_{\frac{n}{N}}(a) = \gamma_{\frac{n}{N}}(b)$

ie the curves $\gamma_s(t)$ are closed.

$\blacksquare$

We've seen in $\mathbb{C}$, $\mathbb{C} - (-\infty, 0]$ or any convex set any 2 closed curves are homotopic or any 2 curves with the same end points are homotopic. This motivates the

Definition An open set $\Omega \subset \mathbb{C}$ is called <u>simply connected</u> if it is <u>connected</u> and if any 2 curves with the same endpoints are homotopic.

$\mathbb{C}$, $\mathbb{C} - (-\infty, 0]$, $D_r(z_0)$ are simply connected

$\mathbb{C} - \{0\}$ is not simply connected

As a corollary of Homotopy thm we have that

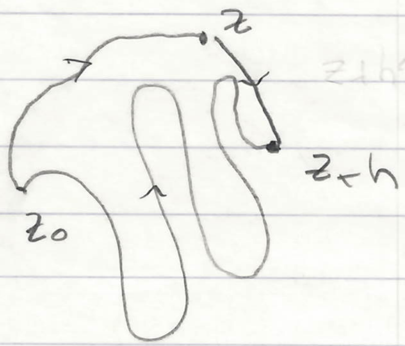
Thm 5.2. Any holomorphic function on a simply connected domain Ω has a primitive. In particular $\int f dz = 0$ for every closed curve $\gamma \in \Omega$. Any 2 primitives differ by a constant

Proof. Fix $z_0 \in \Omega$. For $z \in \Omega$, Ω connected.

Define
$$F(z) = \int_\gamma f(w)\,dw$$

where γ is any curve connecting z_0 to z. The definition is well defined since Ω is simply connected. Hence if $\tilde{\gamma}$ is another curve connecting z_0 to z, then $\gamma \sim \tilde{\gamma}$ and by Homotopy thm

$$\int_\gamma f(\cancel{z}w)\,dw = \int_{\tilde{\gamma}} f(w)\,dw.$$



Choose h small so that the line segment joining z to $z+h$ is in Ω

Then $F(z+h) - F(z) = \int_z^{z+h} f(w)\,dw$

Arguing as in the proof of Thm 2.1, II or using continuity of f as below we get

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$$

which shows F is a primitive of f in Ω ☒.

$$F(z+h) - F(z) = \int_{[z,z+h]} \big(f(w) - f(z) + f(z)\big)\,dw$$

$$= f(z) \underbrace{\int_{[z,z+h]} dw}_{h} + \int_{[z,z+h]} f(w) - f(z)\,dw$$

$$\left| \int_{[z,z+h]} f(w) - f(z)\,dw \right| \le \left(\sup_{w\in[z,z+h]} |f(w)-f(z)| \right) h$$

Hence $\left| \frac{F(z+h)-F(z)}{h} - f(z) \right| \le \sup_{w\in[z,z+h]} |f(w) - f(z)|$

But f is continuous. Hence $\sup_{w\in[z,z+h]} |f(w) - f(z)| \to 0$ as $h \to 0$

The complex logarithm

For $z \in \mathbb{C}$ we want to define a non-zero complex number.

If we write $z = re^{i\theta}$ we want the logarithm to be the inverse of the exponential,

ie $w = \log z$ if $e^{w} = z$

We can set $\log z = \log r + i\theta$

where $\log r$ is the usual logarithm $\log : \mathbb{R}^+ \longrightarrow \mathbb{R}$ of the positive real number r.

The problem is that this is not single valued since θ is only unique up to an integer multiple of 2π.

We want a holomorphic function $\ell : \Omega \to \mathbb{C}$ which satisfy $\exp \circ \ell = \mathrm{id}$ throughout its domain of definition.

Defn: Let $\Omega \subset \mathbb{C}$ be open. A branch of the logarithm $\log_{\Omega}$ on Ω is a holomorfic function s.t $\exp(\log_{\Omega}(z)) = z$ $\forall z \in \Omega$.

Remark ① Since $\exp z \neq 0$ $\forall z \in \mathbb{C}$
such $\log_{\Omega}$ function can exist only if
$0 \notin \Omega$.

② If $\Omega = \mathbb{C} - \{0\}$. Even though

$$\exp : \mathbb{C} \longrightarrow \mathbb{C} - \{0\} \quad \text{is surjective}$$

there is <u>no</u> holomorphic choice of
logarithm in $\mathbb{C} - \{0\}$. Indeed if there
were $f \in \mathcal{H}(\mathbb{C} - \{0\})$ s.t

$$\exp(f(z)) = z \quad \forall z \in \mathbb{C} - \{0\}$$

then differentiating both sides would give

$$f'(z)\exp(f(z)) = 1 \quad \forall z \in \Omega. \quad \text{Hence}$$

$$f'(z) = \frac{1}{z} \quad \forall z \in \mathbb{C} - \{0\}.$$ ie $\frac{1}{z}$ has a
primitive f
but then we would get $\int_\gamma \frac{1}{z}\,dz = 0$
for $\gamma = C_1(0)$ which we know is $2\pi i$

(3) if Ω is open and connected and $l = \log_\Omega : \Omega \longrightarrow \mathbb{C}$ is a logarithm on Ω. Then $\tilde{l} : \Omega \to \mathbb{C}$ is also a logarithm on Ω if and only if

$\tilde{l} = l + 2\pi i n$ for some $n \in \mathbb{Z}$.

Indeed if $\tilde{l}$ is a logarithm function then

$$\exp(\tilde{l}(z)) = z \quad \text{and} \quad \exp(l(z)) = z$$

Hence $\exp\left(\tilde{l}(z) - l(z)\right) = 1 \quad \forall z \in \Omega$

$$\tilde{l}(z) - l(z) \in 2\pi i \mathbb{Z} \quad \forall z \in \Omega.$$

ie $\dfrac{\tilde{l} - l}{2\pi i}$ is a cont., integer-valued function on Ω which is connected

Hence its image under $\dfrac{\tilde{l} - l}{2\pi i}$ is connected and a subset of $\mathbb{Z}$ hence it is a single point n.

Conversely if $\tilde{l} = l + 2\pi i n$ then

$$\exp(\tilde{l}(z)) = \exp(l(z)).\exp(2\pi i n) =$$
$$= \exp(l(z)) = z$$

We have for a simply connected domain $\Omega \subset \mathbb{C} - \{0\}$ the following

Thm 6-1 Let $\Omega \subset \mathbb{C} - \{0\}$ be a simply connected set. Then there exists a branch of the logarithm on Ω

ie a function $F : \Omega \to \mathbb{C}$ s.t

F is holom on Ω and $\exp(F(z)) = z$ $\forall z \in \Omega$.

Proof. Since $0 \notin \Omega$, $\frac{1}{z} \in \mathcal{H}(\Omega)$ and since Ω is simply connected

It has a primitive on Ω.

let $f(z)$ be a primitive of $1/z$

let $G(z) := z \exp(-f(z))$

$$G'(z) = -\underbrace{f'(z)}_{\frac{1}{z}} z \exp(-f(z)) + \exp(-f(z))$$

$$= -\exp(-f(z)) + \exp(-f(z)) = 0$$

Ω connected hence $G(z) = \text{constant} = a$

Since $\exp \neq 0, z \neq 0, a \neq 0$, and $\exists$ b s.t $a = \exp(b)$, and $\exp(f(z)) = \frac{z}{a}$

let $F(z) = f(z) + b$

then $\exp(F(z)) = \underbrace{\exp f(z)}_{=\frac{z}{a}} \cdot \underbrace{\exp(b)}_{a} = z$

and $F(z)$ is a branch of the log on Ω.

Defn let $\Omega := \mathbb{C}^- = \mathbb{C} - (-\infty, 0]$ the principal branch of the logarithm is the unique $\log_\Omega \in \mathcal{H}(\Omega)$ s.t $\log(1) = 0$. Sometimes $\log_{\mathbb{C}^-}$ is also denoted by Log w/ capital L

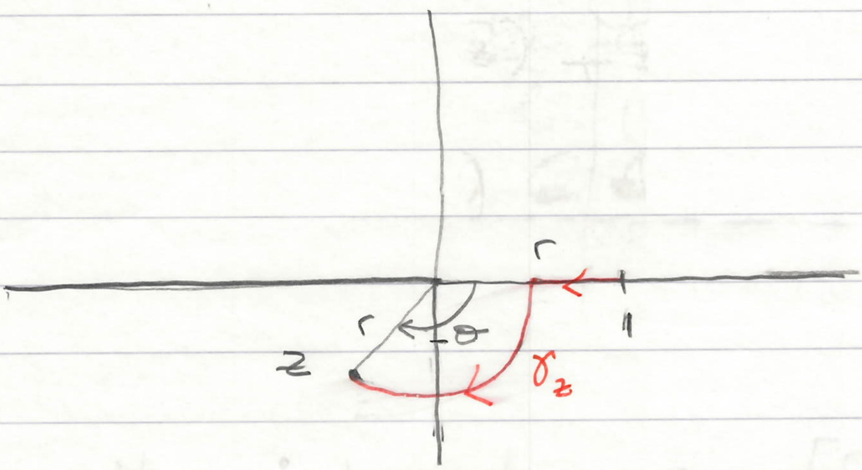
Proposition If $z = re^{i\theta} \in \mathbb{C}^-$ with $r > 0$ $-\pi < \theta < \pi$, then the principal branch of logarithm is given by the formula

$$\log z := \log_{\mathbb{C}^-} z = \log r + i\theta.$$

Proof. let $\log z := \int_{\gamma_z} \frac{dw}{w}$

be a primitive of $\frac{1}{z}$ where we take the path γ_z which starts at 1 and ends at z.

Note $\int_{\gamma_1} \frac{dw}{w} = 0$ hence $\log 1 = 0$.

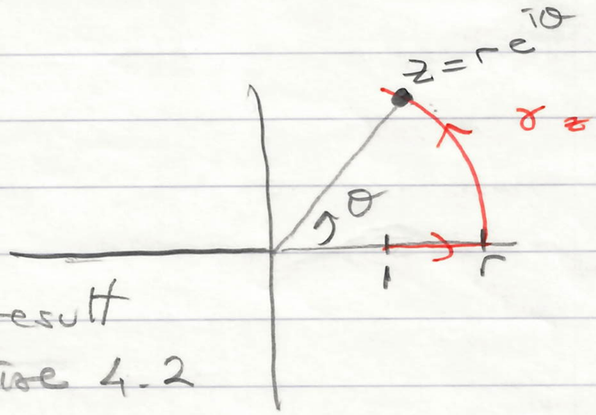


If $z = re^{i\theta}$ w/ $r < 1$ take the path γ_z which goes on the real line from 1 to r then on the circular arc to z.

$$\log z = \underbrace{-\int_r^1 \frac{dx}{x}}_{\text{on the segment}} + \underbrace{\int_0^{-\theta} \frac{-i\cancel{r}\cancel{e^{-it}}}{\cancel{r}\cancel{e^{-it}}}\,dt}_{\substack{\text{on the arc} \\ z = re^{-it} \quad 0 < t < -\theta}}$$

$$= \log r + i\theta$$

If $r > 1$ take the path



Similar calculation gives the result (which was also on exercise 4.2 in the sheet 4) ∎