29.11.2023

Remark (1) The identity $\log z + \log w = \log wz$

does not hold for all $z, w, wz \in \mathbb{C}^-$

If $w = re^{i\alpha}$, $z = se^{i\beta}$ $wz = rse^{i\theta}$

with $\theta, \alpha, \beta \in (-\pi, \pi)$

then $\exists$ $\gamma \in \{-2\pi, 0, 2\pi\}$ s.t.

$$\theta = \alpha + \beta + \gamma$$

Then $\log wz = \log rs + i\theta$

$$= \log r + \log s + i(\alpha + \beta + \gamma)$$

$$= \log r + i\alpha + \log s + i\beta + i\gamma$$

$$= \log w + \log z + i\gamma$$

In particular $\log wz = \log w + \log z$

$$\Leftrightarrow \gamma = 0 \Leftrightarrow \alpha + \beta \in (-\pi, \pi)$$

Since the condition is met whenever $\operatorname{Re} w > 0$, $\operatorname{Re} z > 0$ we have

$\log wz = \log z + \log w$ $\forall w, z \in \mathbb{C}^-$ with $\operatorname{Re} z > 0$, $\operatorname{Re} w > 0$

Remark 2 For the principal branch of log one has the Taylor expansion

$$\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n, \quad |z-1|<1$$

To see this note the derivative of LHS is $\frac{1}{z}$, and RHS's derivative is

$$\sum_{n=1}^{\infty} (-1)^{n-1}(z-1)^{n-1} = \sum_{n=0}^{\infty} (1-z)^n = \frac{1}{1-(1-z)} = \frac{1}{z}$$

for $|z-1|<1$

Hence RHS $= \log z$, LHS $= \sum \frac{(-1)^{n-1}}{n}(z-1)^n$ differ by a constant. Looking at $z=1$ gives both sides are equal to zero. Hence

$$\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n \qquad |z-1|<1.$$

Remark (3)

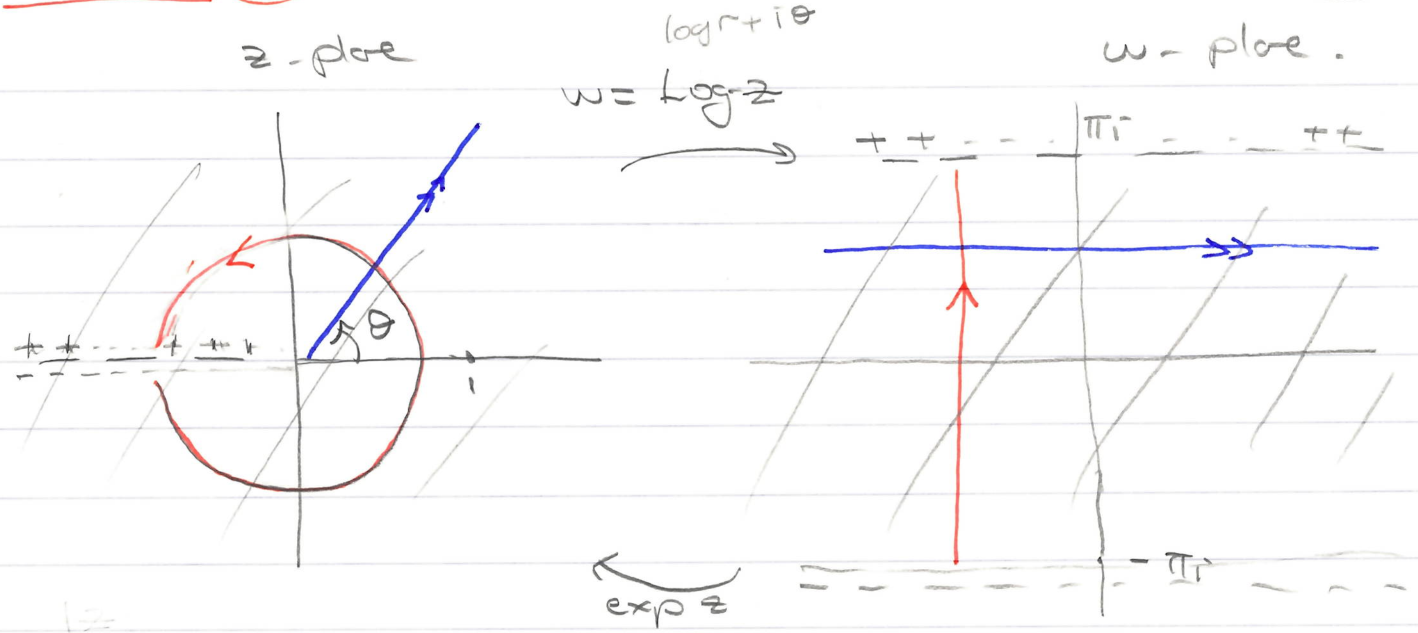
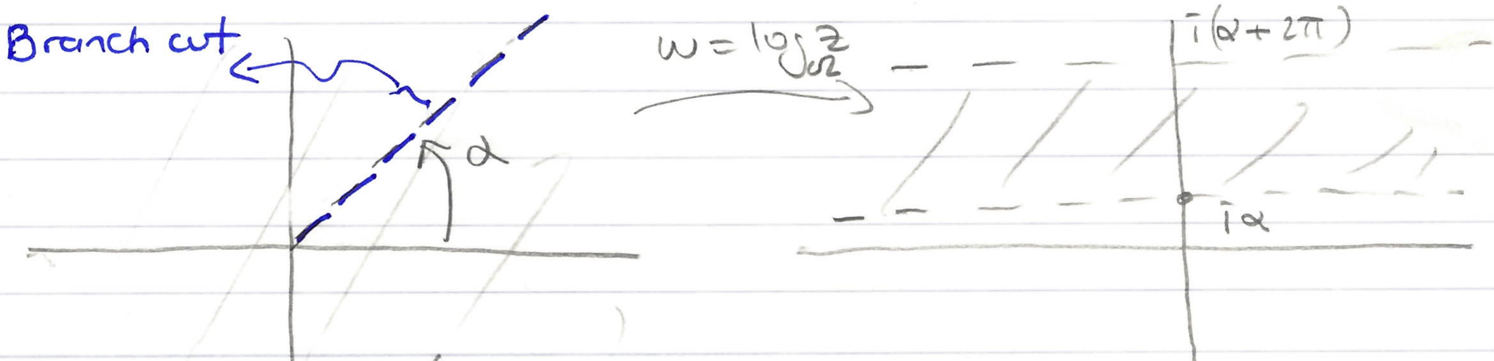


Image of a punctured circle $\{|z| = r \mid -\pi < \arg z < \pi\}$ is the vertical interval $\{\operatorname{Re} w = \log|z|, -\pi < \operatorname{Im} w < \pi\}$

$$\left(\begin{array}{lll} \text{if } r<1 & \text{then} & \operatorname{Re} w < 0 \\ \text{if } R>1 & & \operatorname{Re} w > 0 \end{array}\right)$$

Image of $\{z \mid \operatorname{Arg} z = \theta\}$, a ray from 0 to ∞ is the horizontal line $\{w \mid \operatorname{Im} w = \theta\}$

(4) We can define a holom. branch of logarithm for any $\Omega := \mathbb{C} \setminus (\{z \mid \operatorname{Arg} z = \alpha\} \cup \{0\})$



$$w = \log z = \log|r| + i\theta, \; (r>0, \; \alpha < \theta < \alpha + 2\pi)$$

Remark 5 Let $\Omega \subset \mathbb{C}^*$ be simply connected and $\log_\Omega : \Omega \to \mathbb{C}$ a branch of logarithm.

Let $\alpha \in \mathbb{C}$, $z \in \Omega$, we define

$$\boxed{z^\alpha := \exp(\alpha \log_\Omega z)}.$$

Note this definition depends on the choice of $\log_\Omega$. If we choose $\log_\Omega + 2\pi i k$ instead then

$$\exp(\alpha(\log_\Omega z + 2\pi i k)) = z^\alpha e^{2\pi i k \alpha}$$

If we choose the principal branch of log with $\log 1 = 0$, $\alpha = \frac{1}{m}$

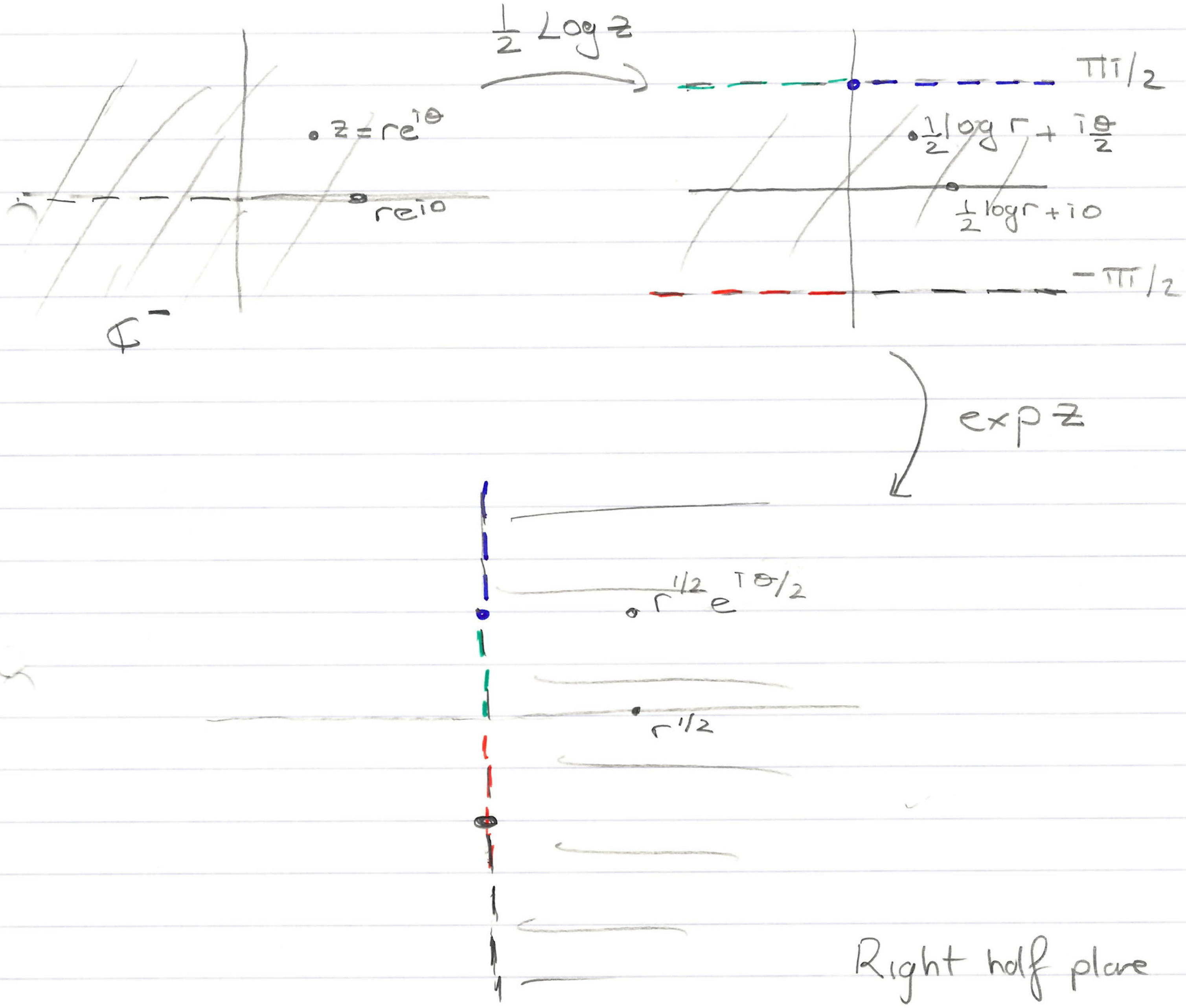
then $z^{1/m} = e^{\frac{1}{m}\log z}$ satisfy

$$(z^{1/m})^m = \exp\left(\tfrac{1}{m}\log z\right) \cdots \exp\left(\tfrac{1}{m}\log z\right)$$

$$= \exp\left(m \tfrac{1}{m} \log z\right) = \exp(\log z) = z.$$

Example: let $\operatorname{Log} z$ be the principal branch of log on $\mathbb{C}^-$

$$z^{1/2} = \exp\left(\tfrac{1}{2}\operatorname{Log} z\right)$$



Note for $z \in \mathbb{R}^+$, $z^{1/2}$ is the usual positive square root.

If we choose $\log_{\mathbb{C}^-} z = \log r + i(\theta + 2k\pi)$

then
$$z^{1/2} = \exp\left(\tfrac{1}{2}\log z\right) = r^{1/2} e^{i\frac{\theta + 2k\pi}{2}}$$
$$= r^{1/2} e^{\frac{i\theta}{2}} \cdot e^{ik\pi}$$
$$= r^{1/2} e^{\frac{i\theta}{2}} (-1)^k = [z^{1/2}](-1)^k$$

∞ly many branches of logarithm yield precisely 2 branches of the square root.
(where we wrote $[z^{1/2}]$ for the principal branch of the square root.

For the choice $\log_{\mathbb{C}^-} z = \log r + i(\theta + 2\pi)$
$(r > 0, \quad \pi < \arg z < 3\pi)$
we have