Finally we have that if $f \in \mathcal{H}(\Omega)$ on a simply connected domain Ω and f is non-vanishing in all of Ω, then

f has a logarithm in Ω. ie $\exists$ a holom g on Ω s-t

$$f(z) = e^{g(z)}$$

The function $g(z)$ is called a logarithm of f and is denoted by $\log f(z)$.

Thm (6-2) If $f \in \mathcal{H}(\Omega)$, non-vanishing in all of Ω, a simply connected domain. Then $\exists$ a holom function $g : \Omega \to \mathbb{C}$, called logarithm of f, such that

$$f(z) = e^{g(z)}$$

Proof Exercise Define g as a primitive of $\frac{f'}{f}$

Cor If $f \in \mathcal{H}(\Omega)$, non-vanishing in all of Ω, simply connected
Then f has a square root in Ω
ie $\exists$ $h : \Omega \to \mathbb{C}$ holom
such that

$$h^2(z) = f(z)$$

Pf Let $h(z) = \exp\left(\frac{1}{2} \log f\right) = \exp\left(\frac{1}{2} g(z)\right)$ from Thm 6-2
then $h^2 = \exp g(z) = f(z)$

Before we move to conformal maps in the next section,
I mention that there are various ways to look at simply connected domains
This is taken up in the book in the Appendix B.

We've seen that if Ω is simply connected (ie any 2 curves in Ω w/ same end points are homotopic) then

$$\int_\gamma f(z)\,dz = 0 \quad \forall \gamma \text{ closed curve in } \Omega \text{ and } f \in \mathcal{H}(\Omega).$$

An open connected region Ω is called holomorphically simply connected if

$\forall \gamma \in \Omega$ closed, $f \in \mathcal{H}(\Omega)$

$$\int_\gamma f(z)\,dz = 0$$

Clearly Ω simply connected $\Rightarrow$ Ω holom. simply connected

in fact the converse is also true. ie we have

<u>Thm</u> A region Ω is holomorphically simply connected $\Leftrightarrow$ Ω is simply connected

The other direction

holom simply connected $\Longrightarrow$ simply connected

uses Riemann mapping thm (which we will see soon)

For bounded domains we also have

<u>Thm</u> If Ω is a bounded region in $\mathbb{C}$ then Ω is simply connected $\Longleftrightarrow$ $\mathbb{C} \setminus \Omega$ is connected

The proof $\underset{\text{bdd}}{\Omega \text{ simply connected}} \Rightarrow \mathbb{C} \setminus \Omega$ connected

uses the notion of winding numbers which we will talk about briefly next since it also leads to the natural generalization of the residue thm.

<u>Rmk.</u> In the above thm Ω bdd is important since the infinite strip is simply connected unbounded, its complement has 2 components

However if the complement is taken in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the conclusion holds if Ω is bdd or not.

Winding numbers

We have seen that if f is holomorphic
$\gamma \subset \Omega$ closed curve, $\gamma_1 \sim_\Omega \gamma_2$ closed curves

then
$$\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$$

We want to understand the integral

$$\int_\gamma f\,dz \qquad \text{for } f \in \mathcal{M}(\Omega).$$

<u>Recall</u> if γ is a circle in Ω then the
$$\int_\gamma f\,dz = 2\pi i \sum \operatorname{Res}_{z_0} f, \quad z_0 \in (\operatorname{int}\gamma \cap S_f)$$

if $\gamma(t) = z_0 + r(t)e^{i\theta(t)}$ in $\Omega \subset \mathbb{C}$
(Ω open)
for some functions r, θ of class C^1
s.t. $r > 0 \quad \forall \ 0 \le t \le 2\pi$
and $r(0) = r(2\pi), \quad \theta(0) = \theta(2\pi)$
the same proof we gave for the residue formula for a circle works and have that

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{\substack{z_0 \in \operatorname{int}(\gamma) \\ z_0 \in S_f}} \operatorname{res}_{z_0}(f)$$

The homotopy thm gives the following first generalization of the residue formula

Proposition Let $\Omega \subset \mathbb{C}$ open, $f \in \mathcal{M}(\Omega)$

Let $V = \Omega - S_f$ so that $f \in \mathcal{H}(V)$

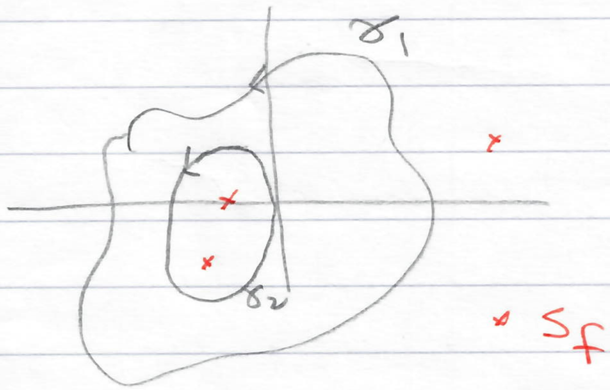
① Let γ_1, γ_2 be closed curves in $V \subseteq \Omega$ which are homotopic in $\underline{\underline{V}}$. Then

$$\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$$

② If γ_2 is a circle (with ccw orientation) then

$$\int_{\gamma_1} f(z) = 2\pi i \sum_{\substack{z_0 \in \operatorname{int}\gamma_2 \\ z_0 \in S_f}} \operatorname{res}_{z_0}(f)$$

Proof.



① This is a special case of Homotopy thm since $f \in \mathcal{H}(V)$ and $\gamma_1 \sim_V \gamma_2$

② Follows from ① and the residue formula.

To look at more general curves we first introduce the winding number of a curve

Defn (Appendix B. p. 347) let $z_0 \in \mathbb{C}$
γ a closed curve in $\mathbb{C}$ such that $z_0 \notin \gamma$
(γ piecewise smooth)
The winding number (or index) of γ around z_0 is defined as

$$w_\gamma(z_0) = \operatorname{ind}_\gamma(z_0) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0}.$$

Remark - To get a feeling for why this is called winding number

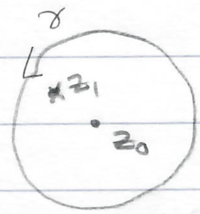
① let's look at the $\gamma(t) = z_0 + re^{it}$

$0 \le t \le 2\pi n$, ie the circle w/ center z_0 traced n times ccw.

Then $$w_\gamma(z_0) = \frac{1}{2\pi i}\int_0^{2\pi n} \frac{ire^{it}}{re^{it}}\,dt = \frac{1}{2\pi}\int_0^{2\pi n} dt = n$$
$$w_\gamma(z_0) = \int_\gamma \frac{dz}{z-z_0}$$

② On the other hand if $\gamma = z_0 + re^{it}$ $0 \le t \le 2\pi$
but we're looking at a pt $z_1 \neq z_0$

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_1}\,dt = \sum_{\tilde z \in (\text{int}\,\gamma \cap S_f)} \text{Res}_{\tilde z} f \quad \text{where } f = \frac{1}{z-z_1}$$



f has only 1 pole, at $z = z_1$ and its residue is 1.

Hence if $z_1 \in \text{int}\,\gamma$ then the integral is 1

and if $z_1 \notin \text{int}\,\gamma$ then $\int_\gamma f = 0$.

So at least when γ is a circle then the integral $\int_\gamma \frac{1}{z-z_1}\,dz$ indeed tells us if γ wraps around z_1 or not.

For a general ~~s~~smooth γ, the following imprecise and really not completely correct argument might give an insight as to why it is called winding number.

For γ smooth
$\gamma : [0,1] \to \mathbb{C}$
$\gamma(0) = \gamma(1)$

$$\int_\gamma \frac{dz}{z-z_0} = \int_0^1 \frac{\gamma'(t)}{\gamma(t)-z_0}\,dt$$

From real analysis we might be tempted to write this last integral

as $\log(\gamma(t) - z_0)\Big|_0^1$ since $\gamma(1) = \gamma(0)$ this would give 0!

But of course this is not correct because $\gamma(t) - z_0$ is complex valued and and if γ wraps around a pt z_0 then we cannot define an analytic branch of $\log(\gamma(t) - z_0)$ on $\mathbb{C} - \{z_0\}$.

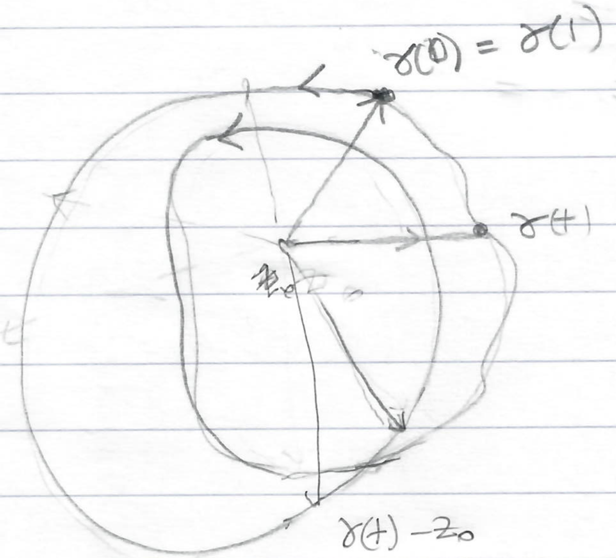
If we think of $\log z = \log|z| + i \arg z$ and recall that the difficulty in defining the logarithm comes from choosing the correct value of the $\arg z$, we can

look at $$\int_\gamma \frac{1}{z - z_0}\,dz = \log(\gamma(1) - z_0) - \log(\gamma(0) - z_0)$$

$$= \log|\gamma(1) - z_0| + i\arg(\gamma(1) - z_0) - \left(\log|\gamma(0) - z_0| + i\arg(\gamma(0) - z_0)\right)$$

$$= i\left(\arg(\gamma(1) - z_0) - \arg(\gamma(0) - z_0)\right)$$

The ambiguity in defining the $\arg(\gamma(t) - z_0)$ for $t=0$, $t=1$ must be an integral multiple of 2π and this integer counts the number of times γ wraps around z_0



We have indeed the following Proposition which shows $w_\gamma(z)$ is always an integer

Prop. (Appendix B, 1-3) Let γ be a closed in $\mathbb{C}$, $\Omega = \mathbb{C} - (\text{image of } \gamma)$ which is open. Then the map

$$w_\gamma : \Omega \longrightarrow \mathbb{C}$$ takes values in

$\mathbb{Z}$ and is continuous. Hence it is constant on any connected subset of Ω. Moreover $w_\gamma(z) = 0$ if $|z|$ is large enough.

Proof. Suppose $\gamma : [a, b] \to \mathbb{C}$ a parametrization of the curve

$G : [a,b] \to \mathbb{C}$

let $G(t) := \displaystyle\int_a^t \frac{\gamma'(s)\,ds}{\gamma(s) - z}$

Note $G(b) = 2\pi i\, w_\gamma(z)$, $G(a) = 0$

Fundamental thm of analysis

$\Rightarrow$ G is continuous (and except possibly at finitely many points), it is differentiable on (a,b) and

$$G'(t) = \frac{\gamma'(t)}{\gamma(t) - z}$$

let $H(t) = (\gamma(t) - z)e^{-G(t)}$

then $H'(t) = \gamma'(t)e^{-G(t)} + \underbrace{(\gamma(t) - z)G'(t)}_{\gamma'(t)} e^{-G(t)}$

$= 0$

Hence H is constant,

ie

$H(t) = (\gamma(t) - z)e^{-G(t)} = c$ for some $c \in \mathbb{C}$

Hence $\gamma(t) - z = ce^{G(t)}$ $\forall t \in [a,b]$

$c = c\underbrace{e^{G(a)}}_{1} = \gamma(a) - z = \gamma(b) - z$

$= ce^{G(b)}$

Hence $e^{G(b)} = 1 \Rightarrow G(b) \in 2\pi i \mathbb{Z}$

$(c \neq 0$, since $\gamma(t) \neq z)$

Since $G(b) = 2\pi i\, \omega_\gamma(z)$ this

shows $\omega_\gamma(z)$ is integer valued

Since $\omega_\gamma(z) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(s)\,ds}{\gamma(s) - z}$ is integral of

a continuous function, it is a continuous function of $z \in \Omega - \{\gamma\}$. Being also integer valued $\omega_\gamma(z)$ is constant in any open connected subset of $\Omega - \{\gamma\}$.

Finally if $M = \sup_{t \in [a,b]} |\gamma(t)|$, and $|z| > M$ then

$$\left|\omega_\gamma(z)\right| = \frac{1}{2\pi}\left|\int_\gamma \frac{dw}{w - z}\right| \leq \frac{1}{2\pi}\frac{\text{length } \gamma}{|z| - M}.$$

Since

$$|w - z| > \big|\,|z| - |w|\,\big| \geq |z| - M$$

Hence $$|\omega_\gamma(z)| \leq \frac{1}{2\pi}\frac{\text{length}(\gamma)}{|z| - M}$$

The RHS goes to zero as $|z| \to \infty$

But then $|\omega_\gamma(z)| < 1$ once $|z|$ is large enough.

But Being an integer this means $\omega_\gamma(z) = 0$ if $|z|$ is large enough

$\blacksquare$

We can now give the general residue formula.

Thm (Residue formula) let $\Omega \subset \mathbb{C}$ simply connected, $f \in \mathcal{M}(\Omega)$

$V = \Omega - S_f$. let γ be a closed curve in V. We have

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_0 \in S_f} \omega_\gamma(z_0)\, \mathrm{res}_{z_0}(f)$$

Proof. For any $z_0 \in S_f$, let P_{z_0} be the principal part of f at z_0

$$P_{z_0}(z) = \sum_{j=1}^{N(z_0)} \frac{a_{-j}(z_0)}{(z-z_0)^j} \quad \text{with some } a_{-j}(z_0) \in \mathbb{C}$$

$N(z_0)$ = order of pole at z_0.

Case 1 S_f is finite. Then

$$\tilde{f} = f - \sum_{z_0 \in S_f} P_{z_0}$$ has removable singularities at $z_0 \in S_f$

Hence has a holom extension to Ω,

Hence

$$\int_\gamma \tilde{f} = 0$$ because Ω is simply connected

Hence $\displaystyle\int_\gamma f(z)\,dz = \sum_{z_0 \in S_f} \int_\gamma P_{z_0}(z)\,dz$

recall $\displaystyle\int_\gamma \frac{dz}{(z-z_0)^j} = 0$ if $j \neq 1$ since

$\dfrac{1}{(z-z_0)^j}$ has a primitive $\dfrac{-1}{(z-z_0)^{j-1}} \cdot \dfrac{1}{(j-1)}$

and γ is closed.

So $\displaystyle\int_\gamma f(z)\,dz = \sum_{z_0 \in S_f} \int_\gamma \frac{a_1(z_0)}{z-z_0}\,dz$

$\displaystyle = \sum_{z_0 \in S_f} a_1(z_0)\, \omega_\gamma(z_0)\, 2\pi i$

$\displaystyle = 2\pi i \sum_{z_0 \in S_f} \left(\mathrm{Res}_{z_0}(f)\right) \omega_\gamma(z_0)$

<u>Case 2</u> S_f is infinite. Pick $R > 0$ s.t $\omega_\gamma(z) = 0$ if $|z| \geq R$

and $\gamma(t)$ is homotopic to the constant curve in $\Omega \cap D_R(0)$. (Since Ω is simply connected $\gamma \sim_\Omega$ (constant curve) which only involves a bounded set.)

Then $S_f \cap D_R(0)$ is finite (Since S_f is a discrete set)

Let $$\tilde{f} = f - \sum_{\substack{z_0 \in S_f \\ |z_0| < R}} P_{z_0} \in \mathcal{H}(\Omega \cap D_R(0))$$

we have $$\int_\gamma \tilde{f} = 0$$ since γ is homotopic to the constant curve in $\Omega \cap D_R(0)$

Hence $$\int_\gamma f = \sum_{\substack{z_0 \in S_f \\ |z_0| < R}} \int_\gamma P_{z_0}$$

$$= 2\pi i \sum_{\substack{z_0 \in S_f \\ |z_0| < R}} \left(\operatorname{Res}_{z_0} f\right) \omega_\gamma(z_0)$$

$$= 2\pi i \sum_{z_0 \in S_f} \left(\operatorname{Res}_{z_0} f\right) \omega_\gamma(z_0)$$

since for $|z_0| \geq R$, $\omega_\gamma(z_0) = 0$

$\blacksquare$

Cauchy Integral formula

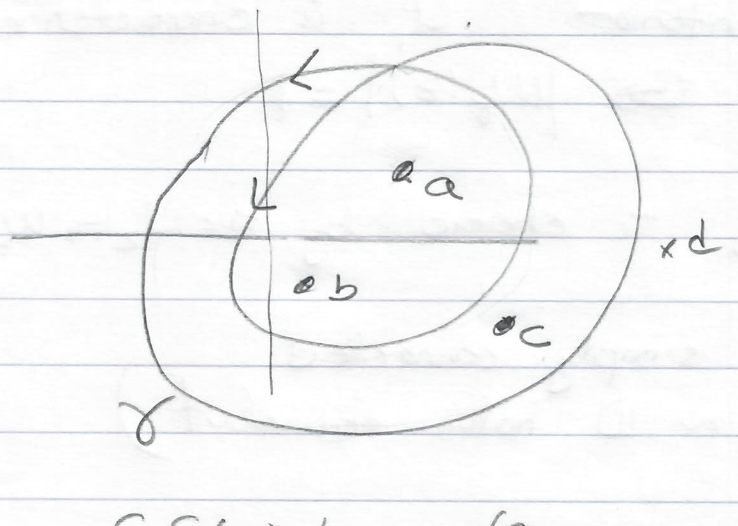
Cor let Ω be simply connected
$f \in \mathcal{H}(\Omega)$, γ a closed curve in Ω

$$\frac{1}{2\pi i}\int \frac{f(w)}{w-z}\,dw = f(z)\,\omega_\gamma(z) \quad \forall z \in \Omega \setminus \gamma$$

Pf This is the generalized residue thm applied to the function $\frac{f(w)}{w-z} = g(w)$

which is meromorphic in Ω except a simple pole at $w=z$ and residue $f(z)$.

Ex



let f be meromorphic except for poles at $z = a, b, c, d$

$$\int_\gamma f(z)\,dz = (\mathrm{Res}_a f)\,2 + (\mathrm{Res}_b f)\,2 + (\mathrm{Res}_c f)\cdot 1$$

§ Conformal Mappings (Chapter 8 in the book).

Motivating question ① Given 2 open sets $U, V \subset \mathbb{C}$, does there exists a holomorphic bijection between them?

ie $f : U \to V$ $f \in \mathcal{H}(U)$
and bijective

We'll see that inverse map

$f^{-1} : V \to U$ is automatically also holomorphic.

(Compare open sets using holom. functions)

② Given an open sset $\Omega \subset \mathbb{C}$ what conditions gurantee that there is a holomorphic bijection from Ω to $\mathbb{D}$?
where
$\mathbb{D}$ = unit disc.

Why $\mathbb{D}$? $\mathbb{D}$ has a very nice geometric structure and most properties of holom functions we developed for $\mathbb{D}$ first. If there is a holom bijection between Ω and $\mathbb{D}$

we can hope to transfer questions about holom functions on Ω to holom functions on $\mathbb{D}$.

① We'll start by examples of such maps and show for example that there is a holom bijection between $\mathbb{D}$ and the upper half plane $\mathbb{H}$.

We can then compose simple maps to get more examples of holomorphic bijections.

② We'll then prove Schwarz lemma which says any $f: \mathbb{D} \to \mathbb{D}$ s.t $f(0)=0$ must satisfy

① $|f(z)| \leq |z| \quad \forall z \in \mathbb{D}$

② If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$ then f is a rotation

③ $|f'(0)| \leq 1$ and if equality holds then f is a rotation

③ Schwarz lemma will then give us all holom bijections of $\mathbb{D}$ to itself.

④ Then we'll get to Riemann-mapping thm which says if $\Omega \neq \mathbb{C}$ or $\emptyset$, and simply connected then there is a holom bijection between Ω and $\mathbb{D}$.

More precisely, for any $z_0 \in \Omega$ $\exists$ a <u>unique</u> $f : \Omega \to \mathbb{D}$ s.t

$$f(z_0) = 0 \quad \text{and} \quad f'(z_0) > 0$$

<u>Rk.</u> Riemann mapping thm says there are only 3 kinds of simply connected domains in $\mathbb{C}$ (up to holom bijections) ϕ, $\mathbb{C}$, $\mathbb{D}$.

Note there can be no holom bijection between $f : \mathbb{C} \to \mathbb{D}$ since then f will be bounded and entire hence by Liouville's f is constant

Note Ω connected is also necessary since $\mathbb{D}$ is connected, same is true for simply connected since if $f : U \to V$ holom bject, U simply connected then so is V.