Defn Let U, V be 2 open sets in $\mathbb{C}$.
An injective holomorphic map $f: U \to V$ is called a conformal map from U to V.
If f is bijective the we say that f is a conformal equivalence or a biholomorphism or a holomorphic isomorphism and U and V are conformally equivalent.
If $U = V$ a conformal equivalence is an automorphism. (Note there is a small difference in the defn of conformal compared to the book. In the book f is taken to be bijective).

Proposition 1.1 (8.1.1) If $f: U \to V$ conformal (ie holomorphic and injective) then $\forall z \in U$ $f'(z) \neq 0$. The inverse of f, which is defined on the image of f is holomorphic ie $f: U \to \text{Image}(f) \subset V$ is a conformal equivalence and $f^{-1}: \text{Image}(f) \to U$ is also a conformal equivalence.

<u>Pf.</u> Suppose f is injective and holom. but on the contrary $\exists\, z_0 \in U$ s.t

$f'(z_0) = 0$. We w.t show that f cannot be injective. Let $g(z) = f(z) - f(z_0)$.

Then $g(z_0) = 0$ and $g'(z_0) = 0$

If $k = \mathrm{ord}_{z_0}(f(z) - f(z_0))$ then by our assumption $k \geq 2$

If $k = \infty$ then $f(z) - f(z_0) \equiv 0$ hence f is constant and cannot be injective.

So can assume $k < \infty$. Then $\exists\, r > 0$ s.t $\forall z \in D_r(z_0)$, we have

$$f(z) - f(z_0) = \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

$$+ G(z)(z - z_0)^{k+1}$$

$$f(z) - f(z_0) = a(z - z_0)^k + G(z)(z - z_0)^{k+1}$$

where $\frac{f^k(z_0)}{k!} = a \neq 0$, $z \in D_r(z_0)$

Note that since zeroes of f' are isolated we can also choose $D_r(z_0)$ s.t $f'(z) \neq 0$ for $z \in D_r^*(z_0)$.

We will use Rouché's thm to show that for a $w \in \mathbb{C}$, $g(z) = f(z) - f(z_0) - w$ has the same number of zeroes as $a(z-z_0)^k - w$ in some disc around z_0.

Since $a(z-z_0)^k = w$ has k solutions, we will have $g(z) = f(z) - f(z_0) - w$ has k zeroes for z sufficiently close to z_0.

Say $z_1, \dots z_k$ are the zeroes of g.

If $w \neq 0$ then these zeroes are not equal to z_0 (Since if $z_k = z_0$, then $0 = g(z_0) = f(z_0) - f(z_0) + w = w \neq 0$.)

Since $f'(z) \neq 0$ for $z \in D_r^*(z_0)$

we have that $g'(z) = f'(z) \neq 0$ for $z \in D_r^*(z_0)$

Hence each zero has order 1 and they are distinct. But that means $\exists$ k distinct $z_1, z_2, \dots z_k$ s.t $f(z_i) = f(z_0) + w$

ie f is not injective.

To show that in some nbhd of z_0, $g(z) = f(z) - f(z_0) - w$ has k zeroes we write for $z \in D_r(z_0)$

$$f(z) - f(z_0) - w = a(z-z_0)^k + G(z)(z-z_0)^{k+1} - w$$

$$= \left(a(z-z_0)^k - w\right) + G(z)(z-z_0)^{k+1}$$

We apply Rouche's thm as follows

Let $C = \sup_{|z-z_0| = \frac{r}{2}} |G(z)|$, C exists since G is continuous

Pick $0 < s < r/2$, $s < 1$ and assume $|w| < |a| \left(\frac{s}{2}\right)^k$. Then on $|z - z_0| = s$,

we have $|a(z-z_0)^k - w| \geq |a| s^k - |a|\left(\frac{s}{2}\right)^k \geq |a| \left(\frac{s}{2}\right)^k$

We also have that

$$\left| G(z)(z-z_0)^{k+1} \right| \leq C s^{k+1}$$

So if $|a|\left(\frac{s}{2}\right)^k > C s^{k+1}$ ie $s < \frac{|a|}{2^k} \frac{1}{C}$

then we can apply Roveche and get that

$g(z) = f(z) - f(z_0) - w$ has the same number of zeroes in $|z - z_0| < s$

as the equation

$$a(z-z_0)^k - w \quad \text{for} \quad |w| < |a|\left(\frac{s}{2}\right)^k$$

(and $s < \min\left(\frac{|a|}{c2^k}, \frac{r}{2}, 1\right)$ as wanted.

Note if $w = re^{i\theta}$ then the zeroes of $a(z-z_0)^k - w$ are at $z_n, n = 0, \ldots k-1$ where

$$z_n - z_0 = \left|\frac{w}{a}\right|^{1/k} e^{i\left(\frac{\theta + 2\pi n}{k}\right)}, \quad n = 0, \ldots k-1$$

But then $|z_n - z_0| = \left|\frac{w}{a}\right|^{1/k} < \frac{s}{2} < s$. Hence all

k roots of $a(z-z_0)^k - w$ are indeed inside $D_s(z_0)$.

The rest is straightforward: $f: U \to f(U)$ is clearly bijective. wlog: assume $f(U) = V$

$f^{-1}: V \to U$ is continuous since

$f: U \to V$ is an open map.

Let $w_0 \in V$, $w \in V$ close to w_0. Write $w = f(z)$

$w_0 = f(z_0)$. If $w \neq w_0$ then

$$\frac{f^{-1}(w) - f(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}}$$

Since $f'(z_0) \neq 0$, and f' continuous, we have

$$\lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{z \to z_0} 1 \Big/ \frac{f(z)-f(z_0)}{z-z_0} = \frac{1}{f'(z_0)}$$

Hence $f^{-1} \in \mathcal{H}(V)$ ∎

Remark ① Prop 1.1 says that If $f: U \to V$ is a conformal equivalence then $f^{-1}: V \to U$ is automatically holomorphic

② The conformal equivalence is an equivalence relation

$U \sim_c U$ since $f: U \to U$ identity
$u \mapsto u$
map is bijective and holomorphic

if $U \sim_c V$ with $f: U \to V$ then

$V \sim_c U$ with $f^{-1}: V \to U$

and if $U \sim_c V$, $V \sim_c W$ w/ $f: U \to V$
$g: V \to W$
then $g \circ f: U \to W$ gives a conformal equivalance between U and W.

③ Conformal equivalence allows the transfer the holom. functions on one set to the holomorphic functions on the other set.

Corollary If $f : U \to V$ is a conformal equivalence then the map

$$T : \mathcal{H}(V) \to \mathcal{H}(U)$$
$$\phi \longrightarrow \phi \circ f$$

(where $\phi : V \to \mathbb{C}$ is a holom. func. on V)

is a linear isomorphism with inverse

$$T^{-1} : \mathcal{H}(U) \to \mathcal{H}(V)$$
$$\varphi \longmapsto \varphi \circ f^{-1}$$

($\varphi : U \to \mathbb{C}$ holom func on U).

ie T is an isom. of vector spaces ie

$$T(a\phi_1 + b\phi_2) = aT(\phi_1) + bT(\phi_2)$$