12-12-23

Important

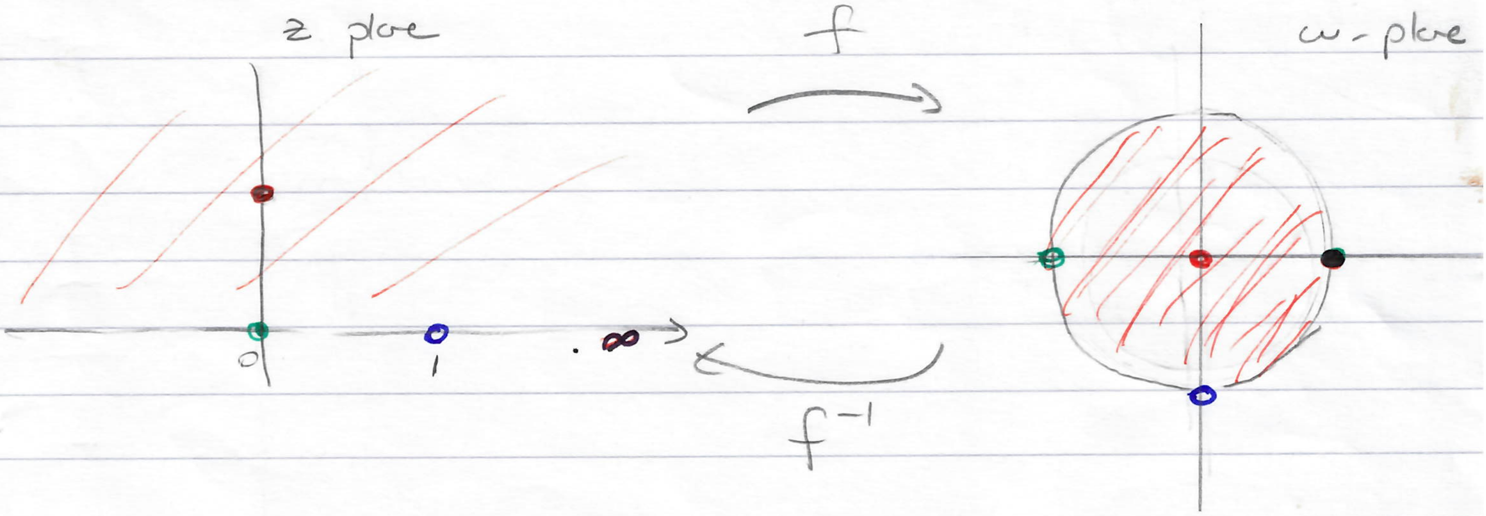
Example (§ 8.1.1) The disc and the UHP.

Let $\mathbb{H} := \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ be the upper half plane.

$\mathbb{D} = D_1(0) := \{ z \in \mathbb{C} \mid |z| < 1 \}$ the unit disc

Then the map $f : \mathbb{H} \to \mathbb{D}$
$z \to \dfrac{z - i}{z + i}$ is

a conformal equivalence, $f^{-1}(w) = i \dfrac{1+w}{1-w}$



Note this example shows that the property that a set is bounded is not preserved under conformal equivalence

Proof. First note that for any $z \in \mathbb{H}$, $|f(z)| = \left|\frac{z-i}{z+i}\right| < 1$

since the distance from z to i is always larger than the distance from z to $-i$, which is in the lower half-plane.

f is clearly holomorphic since $z + i \neq 0 \ \forall z \in \mathbb{H}$

Similarly the map $g(w) = i \frac{1+w}{1-w}$

is holomorphic for $w \in D_1(0)$

to see that $g(w) \in \mathbb{H}$, we look at

the $\operatorname{Im} g(w) = \dfrac{i\left(\frac{1+w}{1+w}\right) - \left(\overline{i\left(\frac{1+w}{1-w}\right)}\right)}{2i}$

$$= \frac{1}{2}\left(\frac{1+w}{1-w} + \frac{1+\overline{w}}{1-\overline{w}}\right) = \frac{1}{2}\left(\frac{(1-\overline{w})(1+w) + (1-w)(1+\overline{w})}{|1-w|^2}\right.$$

$$= \frac{1-|w|^2}{|1+w|^2} > 0 \quad \text{since} \quad |w| < 1$$

Hence g indeed goes from $D_1(0)$ to $\mathbb{H}$.

Finally direct calculation verifies that $f(g(w)) = w$, $(g \circ f)(z) = z$.

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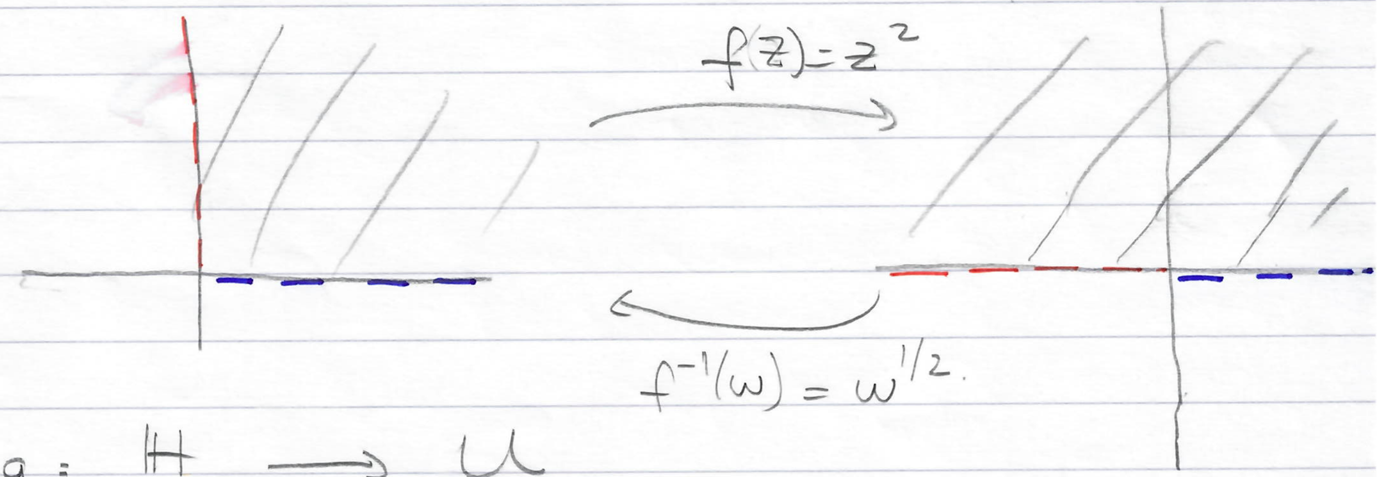
(Note the map takes the real line to the boundary of the disc.

with $f(0) = -1$ $f(1) = -i$, $f(\infty) = 1$

Example The map $z \to z^2$

$$f : U := \{ z \in \mathbb{C} \mid 0 < \arg z < \pi/2 \} \longrightarrow \mathbb{H}$$
$$z \longrightarrow z^2$$

maps the first quadrant to $\mathbb{H}$



$$g : \mathbb{H} \longrightarrow U$$
$$z \longrightarrow z^{1/2} = \exp\left(\tfrac{1}{2} \operatorname{Log} z\right)$$

Note f is injective since if $z_1^2 = z_2^2$ then $z_1 = \pm z_2$ and only one of $z_2, -z_2$ can be in U. Since $z_1, z_2 \in U$ we have that $z_1 = z_2$.

To show surjectivity let $w = re^{i\theta}$, $0 < \theta < \pi$ (2.49)

$w \in \mathbb{H}$, then $z^2 = w$ has 2 solutions

$$z_{1/2} = \pm w^{1/2} = \pm r^{1/2} e^{i\theta/2}$$

and $z = r^{1/2} e^{i\theta/2}$ is in $\mathcal{U}$.

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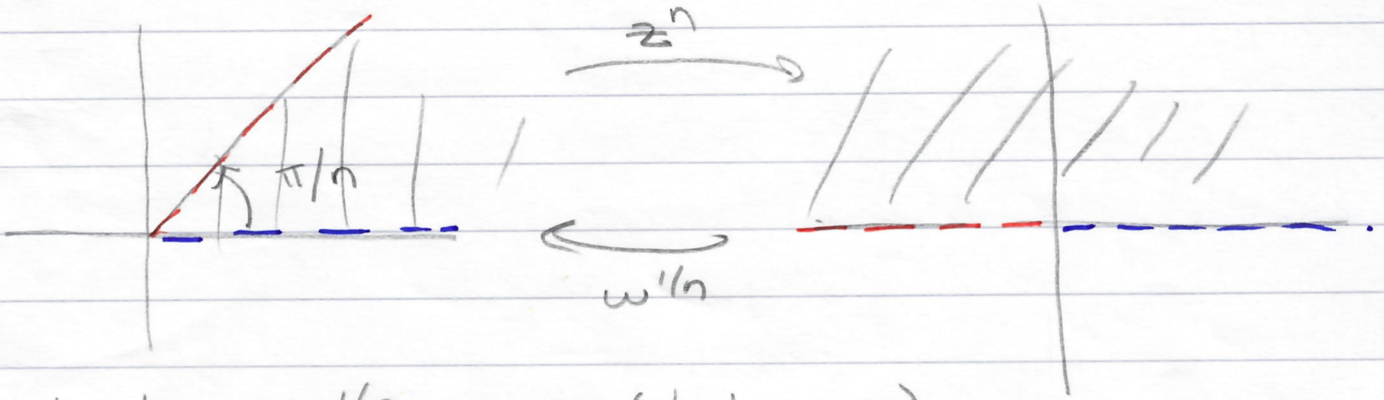
In general the map $z \longrightarrow z^n$ maps a sector $S = \{ z \in \mathbb{C} \mid 0 < \arg(z) < \pi/n \}$



And the $w^{1/n} = \exp(\frac{1}{n} \operatorname{Log} w)$

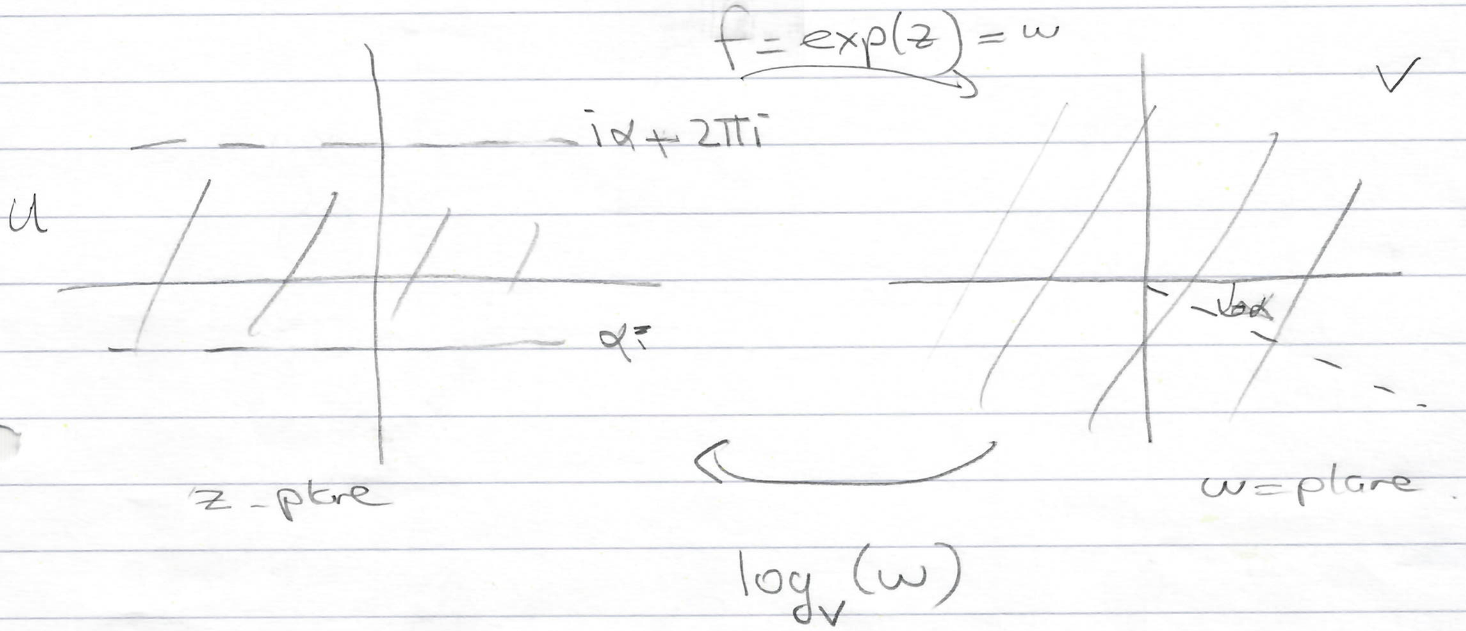
Example The map $f : \mathbb{H} \longrightarrow \mathbb{C}^-$, $z \longrightarrow -z^2$

maps $\mathbb{H}$ to $\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]$

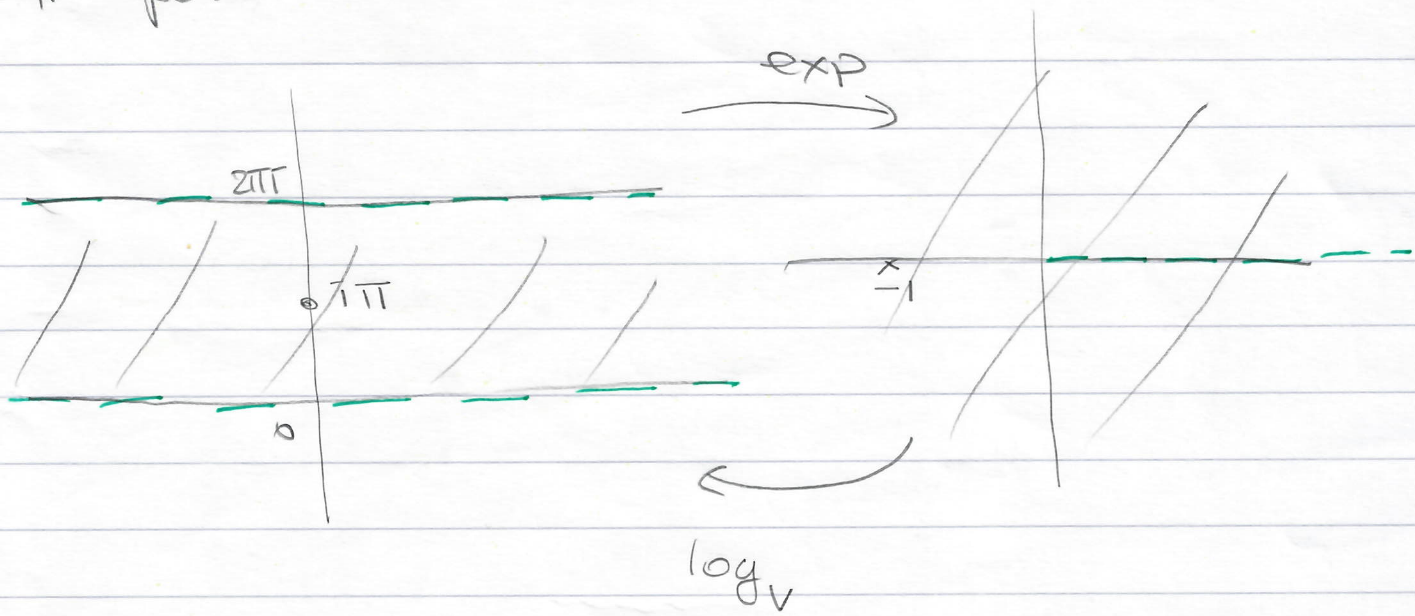
and the map $\tilde{f} : \mathbb{H} \longrightarrow \mathbb{C} \setminus [0, \infty)$, $z \longrightarrow z^2$

maps $\mathbb{H}$ to slit plane cut of the positive reals

Any horizontal strip of length 2π is conformally equivalent to a slit plane



In particular



$\log_V(-1) = i\pi$.

Important non-example $U = \mathbb{C}$, $V = D_1(0)$

Then there is NO biholomorphic map between U and V, since if there were a map
$$f : \mathbb{C} \longrightarrow D_1(0)$$
which is holomorphic. Then f is bounded since $|f(z)| < 1$. Hence by Liouville's thm it is constant, hence is not injective.

Hence $\mathbb{C} \not\simeq_c \mathbb{D}$.

Riemann's thm says that any simply connected domain U which is a proper sset of $\mathbb{C}$, ie $U \neq \emptyset$, and $U \neq \mathbb{C}$ is conformally equivalent to $\mathbb{D}$.

In fact we will prove

Thm (Riemann) (8.3.1) Suppose Ω is proper and simply connected. If $z_0 \in \Omega$ then there is a unique conformal map
$$F : \Omega \longrightarrow \mathbb{D} \quad \text{s.t.} \quad F(z_0) = 0$$
$$F'(z_0) > 0.$$

Cor Any 2 proper simply connected open sset of $\mathbb{C}$ are conformally equivalent

Remark Riemann's mapping thm is remarkable
It classifies all simply connected open ssets $\Omega \subseteq \mathbb{C}$, up to conformal equivalence, there are 3 of them $\emptyset, \mathbb{C}, \mathbb{D}$.
But the proof is not constructive as we'll see. In general it is not easy to find an explicit map.
The rest of the course we'll prove this thm.
The strategy of the proof is as follows.

① **Step 1 - Uniqueness**: This is going to be easy. It boils down to finding all automorphisms of the unit disc.
Since if we have 2 conformal equivalences

$$f_1 : \Omega \longrightarrow \mathbb{D}, \quad f_2 : \Omega \longrightarrow \mathbb{D}$$

then $f_2 \circ f_1^{-1} : \mathbb{D} \longrightarrow \mathbb{D}$ is an autom of $\mathbb{D}$

Step 2 If $\Omega \neq \mathbb{C}$, we'll show there is a conformal map $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$

Hence Ω is conformally equivalent to an open sset of $D_1(0)$. Hence
$\Omega \sim_c f(\Omega) \subset \mathbb{D}_1(0)$

Step 3. Step 2 shows that

The set $\mathcal{F} = \{ f : \Omega \to \mathbb{D} \mid f \text{ conformal}, f(z_0) = 0 \}$

is not empty. We'll see $S := \sup_{f \in \mathcal{F}} |f'(z_0)|$ exists

We'll show that $\exists f \in \mathcal{F}$ s.t

$|f'(z_0)|$ is maximal, ie the supremum

s is taken. This f has "maximal expansion speed"

Step 4. The f we found in step 3 is surjective.

If this is the case writing $f'(z_0) = s e^{i\theta}$

and $g(z) = e^{-i\theta} f$ gives the map we are looking for.

$g : \Omega \to D$ with $g(z_0) = e^{-i\theta} f(z_0) = 0$
$g'(z_0) = s > 0$.

Step 1 Automorphisms of $\mathbb{D}$ and uniqueness

For the automorphisms of $\mathbb{D}$ we have

Thm 2.2 If $f: \mathbb{D} \to \mathbb{D}$ is an automorphism of $\mathbb{D}$, then $\exists$ $\theta \in \mathbb{R}$, and $\alpha \in \mathbb{D}$ s.t

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha} z}$$

Then $f(0) = e^{i\theta}\alpha$, $f'(0) = e^{i\theta}(|\alpha|^2 - 1)$

Conversely all maps of this form are autom of $\mathbb{D}$.

Remark 1 Note an immediate corollary of Thm 2.2 is that only autom of $\mathbb{D}$ that fixes 0 are rotations

Since

$$f(0) = e^{i\theta}\alpha = 0 \Rightarrow \alpha = 0$$

$$\Rightarrow f(z) = -e^{i\theta} z = e^{i\tilde{\theta}} z \text{ for some } \tilde{\theta}$$

2 This thm is enough to prove the uniqueness of conformal equiv.

$$f: \Omega \to \mathbb{D}$$

If f_1, f_2 are 2 such maps, then with $f_1(z_0) = f_2(z_0) = 0$, $f_1'(z_0), f_2'(z_0) > 0$

then $g = f_2 \circ f_1^{-1}: \mathbb{D} \to \mathbb{D}$ autom of $\mathbb{D}$

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Hence $g = e^{i\theta} \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$.

Since $f_1(z_0) = 0$, $f_2(z_0) = 0$

$g(0) = 0$, so $\alpha = 0$ and

$g(z) = -e^{i\theta} z$ for $z \in \mathbb{D}$ and $g'(z) = -e^{i\theta}$

Then

$$-e^{i\theta} = g'(0) = f_2'(f_1^{-1}(0)) . (f_1^{-1})'(0)$$

$$= f_2'(z_0) \cdot \frac{1}{f_1'(z_0)}$$

Hence $\dfrac{f_2'(z_0)}{f_1'(z_0)} = -e^{i\theta} > 0$ (since $f_i'(z_0) > 0$)

$\Rightarrow -e^{i\theta}$ is a positive real number

$\Rightarrow \theta = \pi + 2\pi k.$, $e^{i\theta} = -1$

Since we also have that $\alpha = 0$

$g(z) = z \Rightarrow f_1 = f_2$ $\blacksquare$

The proof of Thm 2.2 uses a simple but important lemma.

Lemma 2.1 (Schwarz's Lemma)

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then

(a) $|f(z)| \leq |z| \quad \forall z \in \mathbb{D}$

(b) If for some $z_0 \neq 0$, we have $|f(z_0)| = |z_0|$ then f is a rotation

(c) $|f'(0)| \leq 1$ and equality holds if and only if f is a rotation. i.e. $\exists\, \theta \in \mathbb{R}$ s.t.
$$f(z) = e^{i\theta} z.$$

Rmk (1) (b) and (c) gives conditions on f so that up to a rotation f is identity map.
Since we assume $f(0) = 0$ the condition in (b) about the $|f(z_0)|$ is true ($\because |f(0)| = |0|$) for $z_0 = 0$ and we cannot conclude from it that f is a rotation. (c) is the condition at 0 that is necessary to conclude f is a rotation (i.e. $|f'(0)| = 1$)

(2) This lemma is once again a statement for holomorphic functions. One cannot conclude for a real diff function $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$ any of a, b, c.

Proof (Schwarz's lemma) It is a consequence of maximum modulus principle

(a) $f(0)=0 \Rightarrow \text{ord}_0 f \geq 1$, we can

define $g(z) = \frac{f(z)}{z}$ for $z \in D_1(0)$

Since $\text{ord}_0 f \geq 1$ and $\text{ord}_0 z = 1$, in fact g has a removable singularity at $z=0$. So

$g \in \mathcal{H}(\mathbb{D})$. Fix $z \in D$

let $0 \leq |z| < r < 1$. For $|\omega| = r$ we have

$$|g(z)| \leq \max_{|\omega|=r} |g(\omega)| = \frac{1}{r} \max_{|\omega|=r} |f(\omega)| \leq \frac{1}{r} \quad \text{since } |f(\omega)|<1$$

By maximum modulus principle

$$|g(z)| \leq \frac{1}{r} \quad \forall z \in \overline{D_r(0)}$$

(holomorphic function g cannot attain a maximum in $D_r(0)$)

This is true $\forall z$, $|z|<r<1$

Letting $r \to 1$ it follows that

$|g(z)| \leq 1$ and hence

$|f(z)| \leq |z| \quad \forall z \in D_1(0)$

Note

(b) (a) gives $\sup_{z \in D_1(0)} |g(z)| \leq 1$

But the assumption $|f(z_0)| = |z_0|$ for some $0 \neq z_0 \in D$ then says g has a local maximum at $z_0 \in D$. By maximum modulus principle

this can only happen if g is constant

Hence $\exists$ $c \in \mathbb{C}$ st $f(z) = z g(z) = cz$

$\forall$ $z \in D_1(0)$. Since $|f(z_0)| = |z_0|$, $|c| = 1$

Hence $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

(c) $g(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0)$

Hence $|f'(0)| = |g(0)| \leq 1$

If $|f'(0)| = 1$, then again 0 is a local maximum of g and we conclude as in (b) to get $f(z) = e^{i\theta} z$ for some θ

We can now give the proof of classification of autom. of $\mathbb{D}$

Proof of Thm 2.2.

First note that any function φ_α of the form $\varphi_\alpha(z) = \dfrac{\alpha - z}{1-\bar{\alpha}z}$ for $\alpha \in \mathbb{C}$ with $|\alpha|<1$

is an autom of $\mathbb{D}$.

Since ① Since $|\alpha|<1$, $1-\bar{\alpha}z \neq 0$ for $|z|<1$

so $\varphi_\alpha \in \mathcal{H}(\mathbb{D})$.

② φ_α is injective: $\varphi_\alpha(z) = \varphi_\alpha(w) \iff$

$$\frac{\alpha - z}{1-\bar{\alpha}z} = \frac{\alpha - w}{1-\bar{\alpha}w} \iff \cancel{\alpha} - |\alpha|^2 w - z + \cancel{\bar{\alpha}zw} = \cancel{\alpha} - |\alpha|^2 z - w + \cancel{\bar{\alpha}zw}$$

$$\iff (1-|\alpha|^2)z = (1-|\alpha|^2)w \iff z = w.$$

Hence φ_α is a conformal map $\varphi_\alpha : \mathbb{D} \to \mathbb{C}$

③ $\varphi_\alpha(\mathbb{D}) \subset \mathbb{D}$: If $|z|=1$ then $z = e^{i\theta}$

$$\varphi_\alpha(e^{i\theta}) = \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \bar{\alpha})} = e^{-i\theta}\left(\frac{\alpha - e^{i\theta}}{e^{-i\theta} - \bar{\alpha}}\right)$$

$$= e^{-i\theta}\frac{w}{-\bar{w}}$$