with $w = \alpha - e^{i\theta}$

Hence $\left|\varphi_\alpha(e^{i\theta})\right| = \left|e^{-i\theta} \cdot \frac{w}{-\overline{w}}\right| = 1$

By max. modulus principle $|\varphi_\alpha(z)| < 1 \quad \forall z \in \mathbb{D}$

(Since $\varphi_\alpha(z)$ is not the constant map, it cannot have a local max. inside $\mathbb{D}$.

(4) $$(\varphi_\alpha \circ \varphi_\alpha)(z) = \frac{\alpha - \frac{\alpha - z}{1-\overline{\alpha}z}}{1 - \overline{\alpha}\left(\frac{\alpha - z}{1 - \overline{\alpha}z}\right)} = \frac{\alpha - |\alpha|^2 z - \alpha + z}{1 - \overline{\alpha}z - |\alpha|^2 + \overline{\alpha}z}$$

$$= \frac{(1-|\alpha|^2)\, z}{1-|\alpha|^2} = z .$$

Hence φ_α is its own inverse.

Clearly any rotation $R(z) = e^{i\theta} z$ is also an autom of $\mathbb{D}$.

Hence $(R \circ \varphi_\alpha)(z) = e^{i\theta} \dfrac{\alpha - z}{1 - \overline{\alpha}z}$ is an autom of $\mathbb{D}$.

Now let f be any autom of $\mathbb{D}$.
Then $\exists$ a unique $\alpha \in \mathbb{D}$ s.t $f(\alpha) = 0$

Consider $g = f \circ \varphi_\alpha$, $g : \mathbb{D} \to \mathbb{D}$.

then $g(0) = f(\alpha) = 0$.

Schwarz lemma (a) applied to g then gives

$$|g(z)| \leq |z| \quad \forall z \in \mathbb{D}$$

Since $g^{-1}(0) = 0$, we can apply Schwarz's lemma to g^{-1} to ~~get~~ get

$$|g^{-1}(w)| \leq |w| \quad \forall w \in \mathbb{D}.$$

Using this for $w = g(z)$ gives

$$|z| = |g^{-1}(g(z))| \leq |g(z)| \quad \forall z \in \mathbb{D}.$$

Combined with $|g(z)| \leq |z|$ we get

that $|g(z)| = |z|$. Once again by

Schwarz's lemma (b) $g(z) = e^{i\theta} z$ for some

$\theta \in \mathbb{R}$. Hence $e^{i\theta} z = (f \circ \varphi_\alpha)(z) = g(z)$

Replacing z with $\varphi_\alpha(z)$ now gives

$$e^{i\theta}\varphi_\alpha(z) = g(\varphi_\alpha(z)) = (f \circ \varphi_\alpha)(\varphi_\alpha(z)) = (f \circ \varphi_\alpha \circ \varphi_\alpha)(z)$$

$$= f((\varphi_\alpha \circ \varphi_\alpha)(z)) = f(z) \quad \text{using } \varphi_\alpha \circ \varphi_\alpha = \mathrm{id}.$$

Rmk. Combining autom. of $\mathbb{D}$ with the Cayley map

$F = \mathbb{H} \longrightarrow \mathbb{D}$ allows one to find all autom of $\mathbb{H}$

$z \longrightarrow \dfrac{z-i}{z+i}$

Thm 2.4 Every autom $g = \mathbb{H} \to \mathbb{H}$ of $\mathbb{H}$ is the form $g(z) = \dfrac{az+b}{cz+d}$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ s.t $ad - bc > 0$

Proof. (Exercise) Read in the book.
Note using the map

$\gamma : Aut(\mathbb{D}) \longrightarrow Aut(\mathbb{H})$
$\varphi \longmapsto F^{-1} \circ \varphi \circ F$

any autom of $\mathbb{D}$ leads to an autom of $\mathbb{H}$.

Moreover γ is an isom with inverse $\gamma^{-1} : Aut(\mathbb{H}) \longrightarrow Aut(\mathbb{D})$
$\beta \longrightarrow F \circ \beta \circ F^{-1}$

Using γ we can pull the autom of $\mathbb{D}$ to autom of $\mathbb{H}$ and show that they are of the above form.

Now we move to Step 2 in the proof of Riemann mapping thm.

Step 2 There is a conformal map

$f : \Omega \to \mathbb{D}$.

(ie Ω is conformally equivalent to a subset of $\mathbb{D}$).

We have the following

Proposition (This is given as step 1 in sec. 3.3 on page 228 in the book)

Let $\Omega \subset \mathbb{C}$, $\emptyset \neq \Omega \neq \mathbb{C}$, simply connected and open. Then there exists a conformal map $f : \Omega \to \mathbb{D}$ s.t

$0 \in f(\Omega)$ ie Ω is conformally equivalent to a subset of $\mathbb{D}$ which contains the origin

Proof. By assumption Ω is proper, hence $\exists$ $\alpha \in \mathbb{C}$ s.t $\alpha \notin \Omega$

By replacing Ω with $\Omega - \alpha := \{ z - \alpha \mid z \in \Omega \}$

we can assume $\alpha = 0$. Hence $\Omega \subset \mathbb{C} - \{0\}$

Since Ω is simply connected, there is

$\log_{\Omega} : \Omega \longrightarrow \mathbb{C}$, $\log_{\Omega} \in \mathcal{H}(\Omega)$

Note $\log_{\Omega}$ is also injective, since if

$\log_{\Omega} z = \log_{\Omega} w$ then exponentiating

both sides we get that $z = \exp(\log_{\Omega} z)$

$= \exp(\log_{\Omega} w) = w$.

Hence $\log_{\Omega}$ is a conformal map

Now let $w \in \Omega$. Then note that for any $z \in \Omega$

$$\log_{\Omega}(z) \neq \log_{\Omega}(w) + 2\pi i$$

Since otherwise, exponentiating we get
$z = \exp(\log_{\Omega} z) = \exp(\log_{\Omega} w) \cdot \exp(2\pi i) = w$
Hence $z = w$ but then $\log_{\Omega}(z) = \log_{\Omega}(w)$
a contradiction

In fact, $\log_{\Omega}(z)$ stays away from $\log_{\Omega}(w) + 2\pi i$

in the sense that $\exists\, \delta > 0$ s.t

$$(*) \quad D_{2\delta}(\log(w) + 2\pi i) \cap \log_{\Omega}(\Omega) = \emptyset$$

Indeed otherwise $\forall\, n > 0$ $(\delta = 1/n)$ we get a sequence $z_n \in \Omega$ s.t

$$\left| \log_\Omega z_n - (\log_\Omega w + 2\pi i) \right| < 1/n$$

Hence $\log_\Omega z_n \longrightarrow \log_\Omega w + 2\pi i$

and exponentiating and using the fact that exp is continuous we get

$z_n \longrightarrow w$ and hence

$\log_\Omega z_n \to \log_\Omega w$ using cont. of $\log_\Omega$

which is a contradiction to $\log_\Omega z_n \to \log_\Omega w + 2\pi i$

Now we can consider the map

$$F : \Omega \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{1}{\log_\Omega z - (\log_\Omega w + 2\pi i)}$$

Since $\log_\Omega$ is injective so is F

Note $F \in \mathcal{H}(\Omega)$ since

$\log z \neq \log w + 2\pi i$ for any $z \in \Omega$.

and hence F is a conformal map

$\circledast \Rightarrow \left|\log_{\Omega} z - (\log(w) + 2\pi i)\right| \geq 2\delta \quad \forall z \in \Omega$

Hence $|F(z) - 0| = \left|\dfrac{1}{\log_{\Omega}(z) - (\log w + 2\pi i)}\right| \leq \dfrac{1}{2\delta} < \dfrac{1}{\delta}$

$\forall z \in \Omega$

Hence $F(\Omega) \subset D_{1/\delta}(0)$. We can now translate and rescale F to obtain a function $f : \Omega \to \mathbb{D}$ which contains origin in its image

Let $f(z) := \dfrac{\delta}{4}\left(F(z) - F(w)\right)$

then $f : \Omega \to \mathbb{C}$ is conformal

we have $f(w) = 0$ and

$$|f(z)| \leq \frac{\delta}{4}\left(\frac{1}{\delta} + \frac{1}{\delta}\right) \leq \frac{1}{2} \quad \forall z \in \Omega$$

Hence $f(\Omega) \subset D_1(0) \quad \forall z \in \Omega$ and

since $f(w) = 0$, $0 \in f(\Omega)$.

$\boxtimes$