Step 3. An extremal Problem.

Let Ω proper, simply connected set of $\mathbb{C}$, $z_0 \in \Omega$

By step 2

$\exists$ $f : \Omega \longrightarrow D_1(0)$ s.t

$$f(z_0) = 0$$

Let $\mathcal{F} := \left\{ f : \Omega \to D_1(0) \;\middle|\; \begin{array}{l} f \text{ conformal} \\ f(z_0) = 0 \end{array} \right\}$

Then $\mathcal{F} \neq \emptyset$

We start by noting the following (See step 2 in section 3.3 in the book)

Lemma The set of values $|f'(z_0)|$, $f \in \mathcal{F}$ is bounded in $[0, \infty)$. Hence

$$\sup_{f \in \mathcal{F}} |f'(z_0)| =: s < \infty$$

Proof Let $\delta > 0$ s.t $D_{2\delta}(z_0) \subset \Omega$

Let $f \in \mathcal{F}$. Cauchy integral formula gives

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_\delta(z_0)} \frac{f(z)}{(z - z_0)^2} \, dz$$

Hence using the standard estimates

$$|f'(z_0)| \leq \frac{1}{2\pi} 2\pi\delta \cdot \frac{\max_{z\in C_\delta}|f(z)|}{\delta^2}$$

$$\leq \frac{1}{\delta} \quad \text{since } |f(z)| \leq 1$$

for all $z \in \Omega$.

Hence $|f'(z_0)|$ is bounded. ☒

The next proposition is the key and says that the supremum

$$s = \sup_{f\in\mathcal{F}} |f'(z_0)| \text{ is taken}$$

Key Proposition: $\exists\ f \in \mathcal{F}$ s.t $|f'(z_0)| = s$

The proof of this Prop uses a compactness argument which we will come to

But we first see why this is key in the sense that it gives the conformal equivalence we're looking for between Ω and $\mathbb{D}$.

(see §3.3, step 3 in the book)
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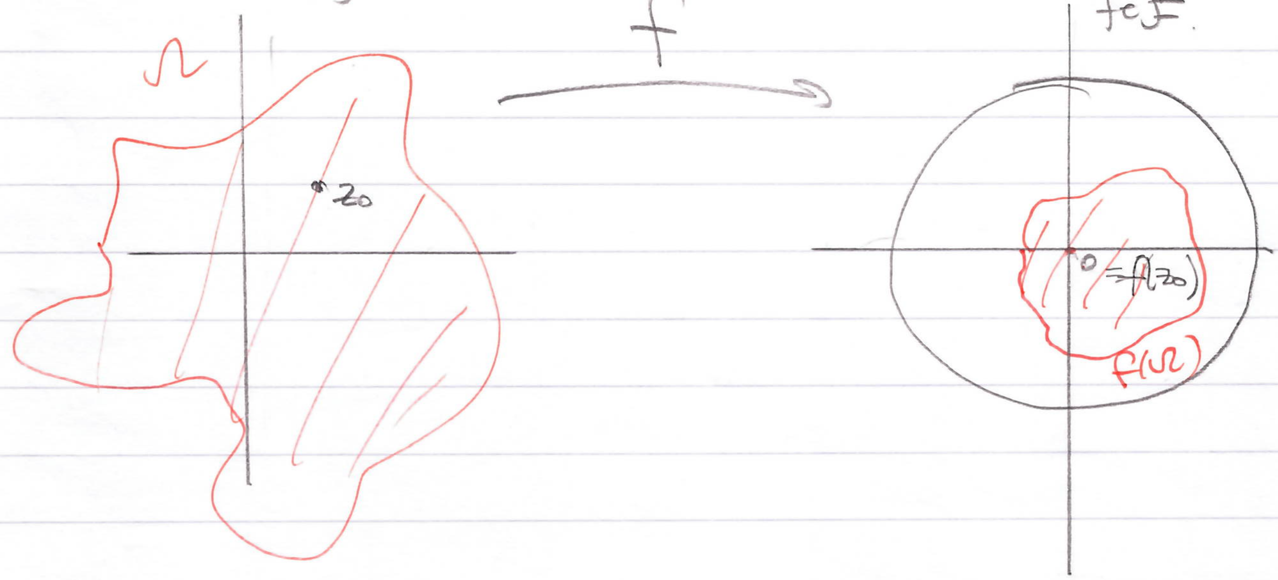
Remark. The step 2 shows that

Ω is conformally equivalent to an open sset of $\mathbb{D}$, which contains 0

why are we looking for an extremal function relazing the extremal value $s := \sup_{f \in \mathcal{F}} \{|f'(z_0)|\}$



We can assume wlog that Ω is an open sset of $\mathbb{D}$ that contains 0. So can assue $z_0 = 0$

We want to conformally strech Ω to fill $\mathbb{D}$.

$$\mathcal{F} = \left\{ f : \Omega \to \mathbb{D} \;\middle|\; \begin{array}{l} f \text{ holom. injective} \\ f(0) = 0 \end{array} \right\}$$

We want to choose a function in $\mathcal{F}$ with "maximal expansion". What does expanding mean.

$f(0) = 0 \Rightarrow f(z) \sim f'(0) z$ for z near 0.

So if $|f'(0)| > 1$ we say f is expanding since the distances between nearby points are expanding

$$|f(z_1) - f(z_2)| \approx |f'(0)||z_1 - z_2| > |z_1 - z_2|.$$

Step 4 f from Key Prop in Step 3 is surjective

Prop. Let $f \in \mathcal{F}$ be such that $|f'(z_0)| = s$. Then f is a conformal equivalence $f : \Omega \to \mathbb{D}$. (ie f is also onto $\mathbb{D}$)

Proof. We want to show that f is surjective. Assume not. Then $\exists\ \alpha \in \mathbb{D}$ which is not in $f(\Omega)$. We will construct $g \in \mathcal{F}$ with $|g'(z_0)| > |f'(z_0)|$ which will be a contradiction to $|f'(z_0)| = s = \sup\limits_{g \in \mathcal{F}} |g'(z_0)|$.

To do this we will use φ_α and the squareroot map.

Let $\varphi = \varphi_\alpha : \mathbb{D} \to \mathbb{D}$ be the autom of $\mathbb{D}$

$$z \longmapsto \frac{\alpha - z}{1 - \bar{\alpha} z}$$

$\varphi_\alpha(0) = \alpha$, $\varphi_\alpha(\alpha) = 0$.

Then $\varphi_\alpha \circ f : \Omega \to D_1(0)$ is conformal

$0 \notin (\varphi_\alpha \circ f)(\Omega)$ since if for some $z \in \Omega$

$(\varphi_\alpha \circ (f(z))) = 0$, then $f(z) = \alpha$ which we assumed is not the case.

Since $0 \notin (\varphi \circ f)(\Omega)$, and Ω is

simply connected, a logarithm, and square root

of of $\varphi \circ f$ exists. ie $\exists$ a holom. map

$\tilde{f} : \Omega \to \mathbb{C}$ s.t $\tilde{f}^2(z) = (\varphi \circ f)(z)$

$\forall z \in \Omega$: [Take $\tilde{g}$ a primitive of $\dfrac{(\varphi_\alpha \circ f)'}{(\varphi_\alpha \circ f)}$

so that $\exp(\tilde{g}(z)) = \varphi_\alpha \circ f$

Then take $\tilde{f} := \exp\left(\frac{1}{2}\tilde{g}(z)\right)$.].

Note $\tilde{f}$ is also injective: If $\tilde{f}(z) = \tilde{f}(w)$

then $(\varphi \circ f)(z) = (\varphi \circ f)(w)$. Since $\varphi \circ f$ is conformal we have $z = w$.

Now $\tilde{f}$ is not yet the function we want

Since $\tilde{f}(z_0) \neq 0$ as $\varphi(f(z_0)) \neq 0$

since $0 \notin (\varphi \circ f)(\Omega)$.

Let $\tilde{f}(z_0) = \beta$ and consider the

autom of $\mathbb{D}$, $\varphi_\beta : \mathbb{D} \to \mathbb{D}$

$$z \to \frac{\beta - z}{1 - \bar{\beta} z}$$

$$\varphi_\beta(\beta) = 0$$

Finally let $g(z) := \varphi_\beta \circ \tilde{f} : \Omega \to \mathbb{D}$

Then $g(z_0) = 0$.

g is holom, since $\varphi_\beta, \tilde{f}$ are
g is injective since $\varphi_\beta, \tilde{f}$ are.

Claim: $|g'(z_0)| > |f'(z_0)|$. This will give the contradiction we are looking for

Recall: we first looked at $\varphi_\alpha \circ f$

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$$\varphi_\alpha \circ f : \Omega \xrightarrow{f} \mathbb{D} \xrightarrow{\varphi_\alpha} \mathbb{D}$$

Then we took the $\sqrt{\ }$ function; call it h

$$h : (\varphi_\alpha \circ f)(\Omega) \longrightarrow \mathbb{D}$$

$$w \longmapsto \exp\left(\tfrac{1}{2} w\right)$$ and composed w/ $\varphi_\alpha \circ f$

$$\tilde{f} = h \circ \varphi_\alpha \circ f : \Omega \longrightarrow \mathbb{D}$$

so that $\tilde{f}^2 = \varphi_\alpha \circ f$

Then we composed with φ_β to get

$$g : \Omega \longrightarrow \mathbb{D}$$

$$g = \varphi_\beta \circ \tilde{f} = \varphi_\beta \circ \underbrace{h \circ \varphi_\alpha \circ f}_{\tilde{f}}.$$

Hence $\varphi_\beta^{-1} \circ g = \tilde{f}$

$$\Rightarrow (\varphi_\beta^{-1} \circ g)^2 = \varphi_\alpha \circ f$$

$$\Rightarrow \varphi_\alpha^{-1} \circ (\varphi_\beta^{-1} \circ g)^2 = f$$

Let $s(z) = z^2$ be the squareing map.

$s : \mathbb{D} \to \mathbb{D}$

Then $f = \underbrace{\varphi_\alpha^{-1} \circ s \circ \varphi_\beta^{-1}}_{:= \Phi} \circ g = \Phi \circ g.$

Note Φ is not injective.

Now $\Phi : \mathbb{D} \to \mathbb{D}$ is holomorphic

$$\Phi(0) = (\varphi_\alpha^{-1} \circ S \circ \varphi_\beta^{-1})(0)$$
$$= \varphi_\alpha^{-1}(\beta^2)$$

But $\beta^2 = (\tilde{f}(z_0))^2 = (\varphi_\alpha \circ f)(z_0)$

Hence $$\Phi(0) = (\varphi_\alpha^{-1} \circ \varphi_\alpha \circ f)(z_0)$$
$$= f(z_0) = 0.$$

Hence we can apply Schwarz's lemma part (c) to get $|\Phi'(0)| < 1$.

(Note $|\Phi'(0)| \neq 1$ since if it were then $\Phi(z) = e^{i\theta} z$ for some θ would mean Φ is injective but Φ is not injective since squaring function is not injective)

Hence using the chain rule applied to $f = \Phi \circ g$ we have

$$f'(z_0) = \Phi'(g(z_0)) \cdot g'(z_0) = \Phi'(0) \cdot g'(z_0)$$

Hence $|f'(z_0)| < |g'(z_0)|$ which is a contradiction ☒

Step 3

§ Proof of the Key Proposition

Existence of the maximum.

① We want to prove the existence of $f \in \mathcal{F}$ s.t $|f'(z_0)| = s = \sup\{g'(z_0) \mid g \in \mathcal{F}\}$.

Recalling the defn of supremum we take a sequence

$$(f_n) \subset \mathcal{F} \quad \text{with} \quad |f_n'(z_0)| \to s$$

we w.t.s.: This sequence has a limit f in $\mathcal{F}$.

Note the proof will not be constructive and will just guarantee the existence of a limit $f \in \mathcal{F}$.

② Recall: We've seen that a sequence of holom. functions that converge unif on compact sets has a holom. limit

But we cannot expect that an arbitrary sequence (f_n) to be unif. conv. on compacts. But may be a subsequence has this property?

Recall in finite dim'l space $\mathbb{R}^n$
every bdd sequence has a convergent subsequence.

So we are looking for an analog of this for $\mathcal{F}$. This is provided by Montel's thm.

<u>Thm 3.3</u> (Montel) Let $\Omega \subset \mathbb{C}$ open. (f_n) a sequence in $\mathcal{H}(\Omega)$. Suppose

that for any compact set $K \subset \Omega$
$\exists$ $M_K \geq 0$ s.t. $|f_n(z)| \leq M_K$ $\forall n \geq 1$ and $z \in K$.

Then $\exists$ subsequence (f_{n_k}) which converges

uniformly on compact ssets of Ω.

(5) In application to Riemann's thm we have a seq $(f_n) \in \mathcal{F}$, so $|f_n(z)| \leq 1$ $\forall z$ and all n (not only compact sets).
Hence we can apply Montel's thm to find a sequence $(f_{n_k}) \in \mathcal{F}$ which converges unf on compacta. and this will give the f we are looking for provided we can show that $f \in \mathcal{F}$

For this we use the following

Proposition Let (f_n) be a sequence in $\mathcal{F}$ and

Suppose that $f_n(z) \to f(z)$ for $z \in \Omega$

unif on any compact set $K \subset \Omega$.

Then either f is constant or $f \in \mathcal{F}$.

and $\lim f_n'(z_0) = f'(z_0)$

Proof Clearly if $f_n \to f$ unif on compacta then $f \in \mathcal{H}(\Omega)$, and $\lim f_n'(z_0) = f'(z_0)$

We need to show that $f(\Omega) \subset \mathbb{D}$ and f is injective or constant.

Since $|f_n(z)| < 1$ we deduce that $|f(z)| \leq 1$

if $|f(z)| = 1$ for some $z \in \Omega$, then z would be a local max of $|f|$ which is impossible by max. mod. principle

So indeed $f : \Omega \to \mathbb{D}$.

What is left to show is that f is injective or constant. For this we have

Lemma 3.5 $\Omega \subset \mathbb{C}$ open connected. $f_n : \Omega \to \mathbb{C}$ conformal. If $(f_n) \to f : \Omega \to \mathbb{C}$ unif on compacta then f is injective or constant

Pf: We'll suppose f is not injective and show that then f is constant

Suppose $\exists\ z_1 \neq z_2 \in \Omega$ s.t $f(z_1) = f(z_2)$

If f is not constant, since zeroes of hol. fncs are isolated, we can find a disc $D_\delta(z_2) \subset \Omega$ $f(z) - f(z_2) \neq 0$ in $D^*_\delta(z_2)$

hence $f(z) - f(z_2) \neq 0 \quad \forall z \in C_{\delta/2}(z_2)$ in particular.

Note this also says that $z_1 \notin C_{\delta/2}(z_2)$. (Since we assumed $f(z_1) = f(z_2)$)

We apply argument principle to the function $f(z) - f(z_1)$ which has a zero namely z_2 in $D_{\delta/2}(z_2)$ to get

$$\frac{1}{2\pi i} \int_{C_{\delta/2}(z_2)} \frac{f'(z)}{f(z) - f(z_1)}\, dz \geq 1$$

We have $f_n \to f$ unif on compacta hence also on $C_{\delta/2}(z_2)$

and $f_n(z) \neq f_n(z_1)$ $\forall n$ and $z \in C_{\delta/2}(z_2)$ since f_n's are injective and $z_1 \notin C_{\delta/2}(z_2)$

Hence $\dfrac{f_n'(z)}{f_n(z) - f_n(z_1)} \to \dfrac{f'(z)}{f(z) - f(z_1)}$ unif on $C_{\delta/2}(z_2)$

Therefore we get

$$\frac{1}{2\pi i}\int_{C_{\delta/2}(z_2)} \frac{f'(z)}{f(z)-f(z_1)} = \lim_{n\to\infty} \underbrace{\frac{1}{2\pi i}\int_{C_{\delta/2}(z_2)} \frac{f_n'(z)}{f_n(z)-f_n(z_1)}}_{=0\ \forall n}$$

Since the integrals on the right counts zeroes of holom function $f_n(z)-f_n(z_1)$ in $C_{\delta/2}(z_2)$ which is none by injectivity of f_n.

But the integral on the left ≥ 1 which is a contradiction. Hence f must be a constant ∎.

Remark

Finally: Note in the case $(f_n) \in \mathcal{F}$ with $\lim |f_n'(z_0)| = s$ and $\lim |f_n'(z_0)| = |f(z_0)|$ we have that

① $f_n'(z_0) \neq 0$ $\forall n$, using Prop 1.1 (Chap 8) and that f_n's are conformal (Hence $f_n'(z) \neq 0$ $\forall z \in \Omega$)

② By defn of supremum, $\exists$ a non-dec. sequence $0 < |f_1'(z_0)| \leq |f_n'(z_0)| < \cdots$ s.t $\lim |f_n'(z_0)| = s > 0$ Hence $f'(z_0) \neq 0$ and f is not constant.