Proof of Montel's thm.

Montel's thm actually consists of 2 parts
① The first part is about complex behaviour of sequences of holomorphic functions which says

(a) A sequence of holomorphic functions which is uniformly bounded on compact sets $K \subset \Omega$ (i.e. $\forall K \subset \Omega$ compact, $\exists M_K > 0$ s.t. $|f_n(z)| \leq M_K \quad \forall z \in K, n \in \mathbb{N}$) is equicontinuous on compact sets

A sequence (f_n) is equicontinuous on a compact set K if $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. if $|z - w| < \delta \quad \forall z, w \in K$ then $|f_n(z) - f_n(w)| < \varepsilon \quad \forall n \in \mathbb{N}$.

This is a complex behaviour in the sense that it is not true for sequences of of real functions. For example $f_n(x) = \sin(nx)$ on $(0,1)$ is uniformly bounded on compact sets but not equicontinuous.

Equicontinuity is a very strong condition and requires uniform continuity uniformly in the family.
The family (f_n) on $[0,1]$ given by $f_n(x) = x^n$

ie: not equicontinuous, even though each
$f_n : [0,1] \to \mathbb{R}$ is unif. cont. on $[0,1]$
$x \to x^n$
The family (f_n) is not equicontinuous

For example take any $0<w<1$ then
$|f_n(1) - f_n(w)| \to 1$ as $n \to \infty$.

(4) The second part is known as
Arzela-Ascoli thm which says
Any family $\mathcal{F}$ of functions which is uniformly bounded and equicontinuous on compact ssets of Ω has a subsequence which converges unif on every compact sset of Ω (The limit need not be in $\mathcal{F}$).

This part belongs to topology / func'l analysis which we'll assume without proof.

ie

Thm Ascoli - Arzela Thm Let $K \subset \mathbb{R}^n$ be compact and $f_n : K \to \mathbb{R}^m$ continuous functions on K.
Suppose (a) $\exists$ $x_0 \in K$ and $M>0$ s.t
$\forall n$ $|f_n(x_0)| \leq M$ (ie $(|f_n(x_0)|)_n$ is bdd in $\mathbb{R}^n$
(b) f_n is equicontinuous. Then
$\exists$ a subsequence (f_{n_k}) which converges unif

on K to some continuous $f : K \to \mathbb{R}^m$.

Assuming Ascoli-Arzela thm, the proof of Montel's thm reduces to proving

that every sequence of holomorphic functions which is unif bounded on compact sets is equicontinuous on compact sets.

This uses Cauchy's integral formula together with the following lemma

Lemma 3.4 (Chap 8) $\Omega \subset \mathbb{C}$ open.
Then $\exists$ compact sets K_ℓ s.t
(a) $K_\ell \subset \operatorname{int} K_{\ell+1}$ $\forall\, \ell = 1, 2, \ldots$

(b) Any compact set $K \subset \Omega$ is contained in K_ℓ for some ℓ.
In particular $\Omega = \bigcup_{\ell=1}^{\infty} K_\ell$

(Such a sequence $\{K_\ell\}_{\ell=1}^{\infty}$ of compact sets of Ω is called an exhaustion.)

Pf. Exercise

Now assuming Ascoli - Arzela thm and lemma 3-4 we can give the proof of Montel's thm.

Let (f_n) be a sequence of holom fncs which are unif bounded on compact sets. We w.t.s show (f_n) is equicontinuous.

Step 1 Let $K \subset \Omega$ be compact
Let $r > 0$ s.t $D_{3r}(z) \subset \Omega$

for $z \in K$ (We can choose r so that $3r$ is less than the distance from K to the boundary of Ω)

Let $z, w \in K$ with $|z-w| < r$. CIF gives

$$f_n(z) - f_n(w) = \frac{1}{2\pi i} \int_\gamma f_n(\zeta) \left[\frac{1}{(\zeta - z)} - \frac{1}{(\zeta - w)} \right] d\zeta$$

where $\gamma = C_{2r}(w)$. On $C_{2r}(w)$, we have

$$\left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right| = \frac{|z-w|}{|\zeta - w||\zeta - z|} \leq \frac{|z-w|}{2r \cdot r}$$

since $|\zeta - z| \geq |\zeta - w| - |w - z| \geq 2r - r = r$
and $|\zeta - w| = 2r$.

Hence using standard estimates for the integral

$$|f_n(z) - f_n(w)| \leq \frac{1}{\cancel{2\pi}} \cdot \cancel{2\pi}(\cancel{2}r)\,|z-w| \cdot \frac{1}{\cancel{2}r^2} M_K$$

$$\leq \frac{|z-w|}{r} M$$

Since $\forall \xi, \forall n$ $|f_n(\xi)| \leq M$ where M is the uniform bound for all $f_n \in \mathcal{F}$ $\overline{D_{2\delta}(w)}$.

So for any $\varepsilon > 0$ we will get

$$|f_n(z) - f_n(w)| < \varepsilon \quad \forall n, \ \forall z, w \in K$$

as soon as $|z-w| \leq \min\left(r, \frac{\varepsilon r}{M}\right)$

<u>Step 2</u> To extract a subsequence which converges unif. on <u>all</u> compact sets we use a standard trick, called the "diagonal argument".

Let K_ℓ be the sequence of compact sets given by the last Lemma 3.4

By Step 1, and Ascoli–Arzela there is a subseq. of (f_n) converging uniformly on K_1, say $(f_n)_{n \in L_1}$

Then $\exists$ a subseq of $(f_n)_{n \in L_1}$ conv.

$L_1 \subset \mathbb{N}$ infinite

uniformly on K_2, hence on K_1, K_2

Say $(f_n)_{n \in L_2}$ $L_2 \subset L_1 \subset \mathbb{N}$ infinite

Inductively we get a subsequence

$(f_n)_{n \in L_k}$ conv. unif on $K_1, K_2 .. K_k$

$L_k \subset L_{k-1} \subset .. \subset L_1 \subset \mathbb{N}$.

Now let $n_1 = \min \{ n \mid n \in L_1 \}$

$n_2 = \min \{ n \mid n \in L_2 \setminus \{n_1\} \} \in L_2 \subset L_1$
$n_3 = \min \{ n \mid n \in L_3 \setminus \{n_1, n_2\} \} \in L_3 \subset L_2 \subset L_1$
$\vdots$

We get $n_1 < n_2 < \dots$ with $n_k \in L_k \subset L_{k-1} \cdots \subset L_1$

Note $L = \{ n_1, n_2, \dots \}$ has the property that $L \setminus L_k$ is finite for each k.

For each k,
$(f_n)_{n \in L} = (f_{n_j})$ is upto finitely many terms (which has no effect on convergence) is a subsequence of $(f_n)_{n \in L_k}$.

Hence (f_{n_j}) conv. unif. on every K_k.

Since any compact set K is contained in K_k for some k we're done $\boxtimes$