What we've shown can be summarized in

Prop (Prop 2.3 in the book) If f is holom. at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0)$$
$$= 2\frac{\partial u}{\partial z}(z_0)$$

If we write $\tilde{f}(x,y) = f(z)$, then $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable then

$$J_{\tilde{f}}(x_0,y_0) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix}$$

and $\det J_{\tilde{f}} = |f'(z_0)|^2$
$$= u_x^2 + u_y^2$$

Proof $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv)$

$$= \frac{1}{2}\left[u_x + i v_x + i u_y - v_y\right]$$

$$= \frac{1}{2}(u_x - v_y) + i(v_x + u_y) = 0$$
using CR

$$\frac{(A)+(B)}{2} \Rightarrow f'(z_0) = \frac{1}{2}\left[f_x(z_0) - i f_y(z_0)\right]$$
$$= \frac{\partial f}{\partial z}(z_0)$$

CR also gives $\frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0) =$

$$= u_x - i u_y \underset{CR}{=} u_x + i v_x$$

If $z_0 = x_0 + iy \in U$, $h = h_1 + h_2 i \in \mathbb{C}$

then f hol. at z_0 means

$$f(z_0 + h) = f(z_0) + f'(z_0)h + h\,\varepsilon(h)$$

with $\lim_{h \to 0} \varepsilon(h) = 0$

If $f'(z_0) = a + ib$ then

$$\boxed{\begin{aligned} f'(z_0) &= u_x + iv_x \\ &= v_y - iu_y \end{aligned}}$$

$$\begin{aligned} f'(z_0)h &= (a+ib)(h_1 + h_2 i) \\ &= ah_1 - bh_2 + i(bh_1 + ah_2) \end{aligned}$$

Hence if we write $\tilde{f}(x,y) = f(z)$, $H = (h_1, h_2)$ we have that

$$\frac{\left|\tilde{f}\left((x_0, y_0) + (h_1, h_2)\right) - \tilde{f}(x_0, y_0) - \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right|}{|H|} \to 0$$

as $|H| \to 0$

This means $\tilde{f} = \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with

$$J_{\tilde{f}}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Using $a = u_x = v_y$, $b = v_x = -u_y$ we get

$$J_{\tilde{f}} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}, \quad \det J_{\tilde{f}} = u_x^2 + v_x^2 = |f'(z_0)|^2$$

The previous prop. shows that
f holom $\Rightarrow \dfrac{\partial f}{\partial \bar{z}} = 0$ (CR. equations)

We also have the following converse

Thm 2.4 Suppose $f = u + iv$, $f : \Omega \to \mathbb{C}$
Ω open set. If $u, v \in C^1$ (continously differentiable) and satisfy CR equations on Ω, then f is holomorphic on Ω and $f'(z) = \partial f / \partial z$.

Proof Let $z_0 = (x_0, y_0) \in \Omega$, $h = (h_1, h_2) \in \mathbb{C}$
$u, v \in C^1$ imply that

$$u(z_0 + h) - u(z_0) = \partial_x u(z_0)\, h_1 + \partial_y u(z_0) h_2 + |h|\, \varepsilon_1(h)$$

with $\varepsilon_1(h) \to 0$ as $h \to 0$
Similarly

$$v(z_0 + h) - v(z_0) = (\partial_x v)\, h_1 + \partial_y v\, h_2 + |h|\, \varepsilon_2(h)$$

with $\lim \varepsilon_2(h) \to 0$.

$$\begin{aligned} f(z_0 + h) - f(z_0) &= (u + iv)(z_0 + h) - (u + iv)(z_0) \\ &= (\partial_x u + i \partial_x v)\, h_1 + (\partial_y u + i \partial_y v)\, h_2 \\ &\quad + |h|\, \varepsilon(h) \end{aligned}$$

where $\varepsilon(h) = (\varepsilon_1 + \varepsilon_2)(h) \to 0$ as $|h| \to 0$

$$f(z_0+h) - f(z_0) = (\partial_x u - i\partial_y u)\, h_1 +$$

using CR.

$$(\partial_y u + i\partial_x u)\, h_2 + \varepsilon(h)|h|$$

$$= (\partial_x u - i\partial_y u)(h_1 + h_2 i) + \varepsilon(h)|h|$$

Hence $f(z_0+h) - f(z_0) = (\partial_x u - i\partial_y u)\, h + \varepsilon(h)\,|h|$

with $\varepsilon(h) \to 0$

which says $\dfrac{f(z_0+h) - f(z_0)}{h} \to \partial_x u - i\partial_y u$

as $h \to 0$

Hence $f'(z_0)$ exists and equal to

$$\partial_x u - i\partial_y u = 2\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}$$

Example: let $f(z) = \overbrace{x^2+y^2}^{u} + \overbrace{2i\,xy}^{iv}$

Then $\partial_x u(x,y) = 2x \qquad \partial_x v = 2y$

$\partial_y u(x,y) = 2y \qquad \partial_y v = 2x$

$\partial_x u = \partial_y v \quad \forall z \in \mathbb{C}$

$2x = 2x$

$2y = \partial_y u = -\partial_x v = -2y$

only if $y = 0$

Hence $f(z)$ is holomorphic for points only on the real axis. And for these points

$$f'(x_0) = \partial_x u(x_0) + i\,\partial_x v(x_0) = 2x_0$$

A quick summary: Ω open sset of $\mathbb{C}$

① $f: \Omega \to \mathbb{C}$, $f(z) = u + iv$

f is holom on Ω $\Rightarrow$ u, v satisfy CR eqns

$$u_x = v_y$$
$$u_y = -v_x$$

$$f'(z) = u_x + i u_y$$

② If $u, v \in C^1$ and satisfy CR-eqns then $f = u + iv$ is holomorphic.

③ If we write $\tilde{f}(x,y) = f(z)$, for $f: \Omega \to \mathbb{C}$ identifying $\mathbb{C}$ with $\mathbb{R}^2$ holomorphic then $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with

$$J_{\tilde{f}}(z_0) = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}, \quad \det J_{\tilde{f}} = |f'(z_0)|^2$$

Remark A matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ defines a linear map $\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

which preserves angles and orientation (ie it is a rotation and a dilation).
If $a + bi \neq 0$, $a + ib = |a+ib| e^{i\theta}$, then it is a rotation by the angle θ, and dilation by $|a+ib|$

Our next result gives important examples of holomorphic functions

§2.3 Power series

Recall a power series is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n, \qquad a_n \in \mathbb{C}$$

Thm (Thm 2.5) Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series.

Then $\exists R \in \mathbb{R}$, $0 \leq R \leq \infty$ such that

(i) if $|z| < R$ the series converges absolutely

(ii) if $|z| > R$ " " diverges.

Moreover with the convention that $1/0 = \infty$ and $1/\infty = 0$, R is given by

$$1/R = \limsup |a_n|^{1/n}$$

R is called the radius of convergence

$D_R(0) := \{ z \in \mathbb{C} \mid |z| < R \}$ disc of convergence

Proof of thm 2.5 Exercise
(same as in real analysis)

Important example of a power series is the complex exponential function

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$ It converges abs $\forall z \in \mathbb{C}$.

Also, $$|e^z| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty$$

Hence e^z is conv. uniformly on compact ssets of $\mathbb{C}$

The following thm shows that e^z in particular and power series in general give examples of holomorphic functions in their disc of convergence.

Thm 2.6 The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence and

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

$f'(z)$ has the same radius of convergence as f.

Proof is similar to the one in real variables but we'll repeat it here since it is an important Thm.

Let R be the radius of convergence of $f(z)$

Since

$$\lim_{n\to\infty} n^{1/n} = 1$$

$$\limsup |a_n n|^{1/n} = \limsup |a_n|^{1/n} = R$$

Hence $\sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence. Repeated application shows that the sum $\sum_{n=0}^{\infty} n(n-1)\cdot(n-k)\, a_n z^{n-k}$ has radius of convergence R for any k

Let $z \in \mathbb{C}$, $|z| < R$ choose δ s.t

$|z| + \delta < R$ eg can take $\delta = \frac{R - |z|}{2}$

Let $h \in \mathbb{C}$, $|h| < \delta$, w.t.s.

$$\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} \right| \to 0 \text{ as } h \to 0.$$

$$\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} \right|$$

$$= \left| \sum_{n=0}^{\infty} \left(\frac{a_n (z+h)^n - a_n z^n}{h} - n a_n z^{n-1} \right) \right|$$

$$\leq \sum_{n=0}^{\infty} |a_n| \left| \frac{1}{h}\left(\sum_{k=0}^{n} \binom{n}{k} h^k z^{n-k} - z^n\right) - n z^{n-1} \right|$$

$$= \sum_{n=2}^{\infty} |a_n| \left| \sum_{k=2}^{n} \binom{n}{k} h^{k-1} z^{n-k} \right|$$

$$\leq \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \binom{n}{k} |h|^{k-1} |z|^{n-k}$$

$$\leq \sum_{n=2}^{\infty} |a_n| \; n(n-1) \sum_{k=2}^{n} \binom{n-2}{k-2} |h|^{k-2} |z|^{n-k} |h|$$

Using, for $k \geq 0$

$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

$= \frac{n(n-1)}{k(k-1)}\binom{n-2}{k-2}$

$\leq n(n-1)\binom{n-2}{k-2}$

$(k \geq 2)$

$$\leq \sum_{n=2}^{\infty} |a_n| \, n(n-1) \, (|z|+|h|)^{n-2} |h|$$

$$\leq \underbrace{\left(\sum_{n=2}^{\infty} |a_n| \, n(n-1)\left(\frac{R+|z|}{2}\right)^{n-2}\right)}_{\text{indep of } h} |h|$$

since $|h| < \frac{R-|z|}{2}$

As $h \to 0$, RHS $\to 0$

Hence $\frac{f(z+h)-f(z)}{h} \xrightarrow[h \to 0]{} \sum_{n=0}^{\infty} n a_n z^{n-1}$

∎

Examples

① $\exp : \mathbb{C} \to \mathbb{C}$

$$z \to \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{conv. } \forall z$$

As such is holom. on all $\mathbb{C}$.

$$\exp(z)' = \sum_{n=0}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

We will write e^z instead of $\exp(z)$ mostly.

② Trigonometric functions

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} \quad , \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{z}^{2n}}{2n!} \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos i = \frac{e^{i^2} + e^{-i^2}}{2} = \frac{e^{-1} + e}{2} = \frac{e^2 + 1}{2e}$$

$$\sin i = \frac{e^{i^2} - e^{-i^2}}{2i} = i\left(\frac{e^2 - 1}{2e}\right)$$

③ $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ has conv. radius 1.

converges $\forall z, |z| \leq 1$ since $\sum \frac{1}{n^2} < \infty$

④ Geometric series : $\sum_{n=0}^{\infty} z^n$ converges for $|z| < 1$

⑤ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$ conv for $|z| < 1$

for $z = 1$ conv (Leibniz' criteria).

$z = -1$ div Harmonic series

§3 Complex line integrals (Integrals along curves).

We start by recalling the main definitions and properties of curves.

Defn ① A <u>parametrized curve</u> in $\mathbb{C}$ is a continuous function $\gamma : [a,b] \to \mathbb{C}$, where $[a,b]$ is a closed interval of real numbers.

② A <u>smooth curve</u> is a curve $\gamma : [a,b] \to \mathbb{C}$
$\gamma(t) = x(t) + iy(t)$ if its derivative
$\gamma'(t) = x'(t) + iy'(t)$ exists $\forall t \in [a,b]$
and if $\gamma'(t)$ is continuous on $[a,b]$.
and $\gamma'(t) \neq 0$ for $t \in [a,b]$.

Here
$$\gamma'(a) := \lim_{\substack{h \to 0 \\ h > 0}} \frac{\gamma(a+h) - \gamma(a)}{h},$$

$$\gamma'(b) := \lim_{\substack{h \to 0 \\ h < 0}} \frac{\gamma(b+h) - \gamma(b)}{h}$$
are the right and left hand derivatives resp.

③ A <u>piecewise smooth</u> curve is a curve $\gamma : [a,b] \to \mathbb{C}$, γ is cont. on $[a,b]$
and $\exists$ points $a = a_0 < a_1 < \cdots < a_n = b$
s.t. $\gamma(t)$ is smooth on each interval $[a_k, a_{k+1}]$.