(4) A closed curve is a curve $\gamma : [a,b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$

(5) A curve is simple if it is not self intersecting ie $\gamma(t) \neq \gamma(s)$ unless $s=t$ or $s=a$ and $t=b$.

Rmk For us in this course the curves will always be piecewise smooth. From now on when we say a curve we mean a piecewise smooth one even if I forget to write.

(6) $\tilde{\gamma} : [c,d] \to \mathbb{C}$ is called reparametrization of $\gamma : [a,b] \to \mathbb{C}$ if there exists a continously differentiable function

$\sigma : [c,d] \to [a,b]$ which is bijective with $\sigma'(t) > 0$ $\forall t$, and

$\tilde{\gamma} = \gamma \circ \sigma$. $\sigma'(t) > 0$ means the orientation is preserved.

(γ, $\tilde{\gamma}$ represents the same geometric object with diffeent parametrizations.

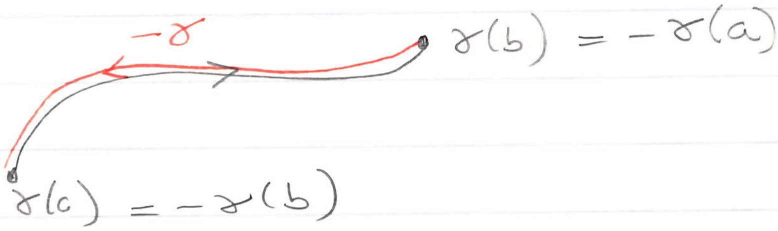
Remark
We will often work with a particular parametrization since most important notions will be independent of parametrization (for example path integrals) Because of this independence, we often describe curves by drawing them as geometric objects in the plane

There are 2 elementary methods to modify or combine paths to obtain new paths.

① If $\gamma : [a, b] \to \mathbb{C}$ is a path
$t \to \gamma(t)$

the reverse path $-\gamma$ (or γ^{-})
is the path $-\gamma : [a, b] \to \mathbb{C}$
$t \to \gamma(b + a - t)$

i.e. $(-\gamma)(t) = \gamma(a + b - t)$



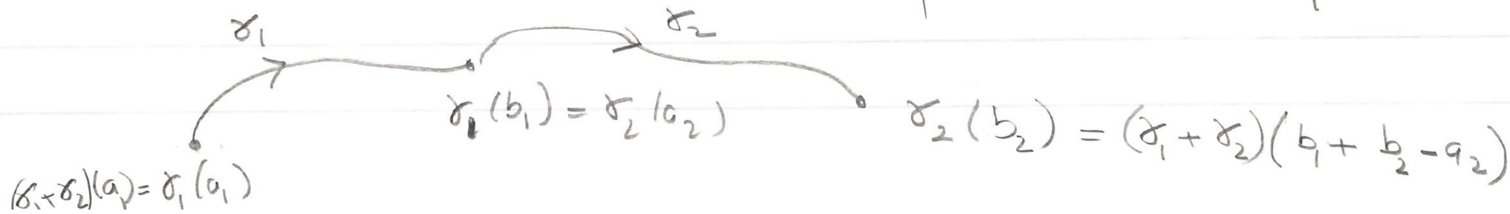
② If $\gamma_1 : [a_1, b_1] \to \mathbb{C}$, $\gamma_2 : [a_2, b_2] \to \mathbb{C}$

are 2 path s.t $\gamma_1(b_1) = \gamma_2(a_2)$ then

the concatenetaion or sum of the paths γ_1, γ_2

is a path $\gamma_1 + \gamma_2 : [a_1, b_1 + b_2 - a_2] \to \mathbb{C}$

defined as $[\gamma_1 + \gamma_2](t) := \begin{cases} \gamma_1(t) & \text{if } a_1 \le t \le b_1 \\ \gamma_2(t - b_1 + a_2) & b_1 \le t \le b_1 + b_2 - \end{cases}$



Examples (1) Given 2 points $z_1, z_2 \in \mathbb{C}$ the path

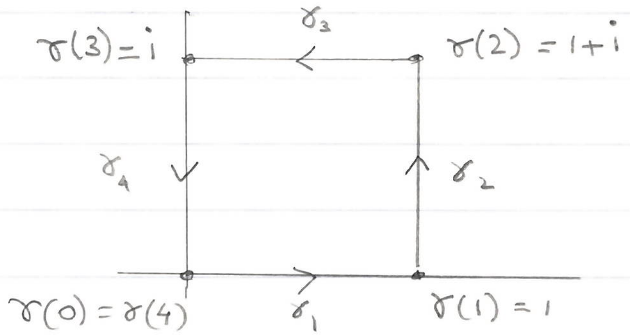
$$\gamma : [0,1] \to \mathbb{C}$$
$$t \mapsto (1-t)z_1 + z_2 t$$

is the parametrization of the line segment between z_1 and z_2

smooth
simple
not closed.

(2) $\gamma : [0,4] \to \mathbb{C}$

$$\gamma(t) = \begin{cases} t & \text{if } 0 \le t \le 1 \\ 1 + i(t-1) & \text{if } 1 \le t \le 2 \\ (3-t) + i & \text{if } 2 \le t \le 3 \\ i(4-t) & \text{if } 3 \le t \le 4 \end{cases}$$



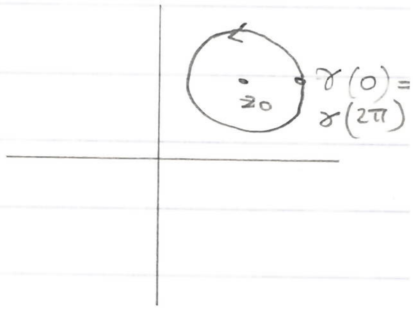
piecewise smooth
simple closed

γ is sum of 4 paths: $\gamma_1 : [0,1] \to \mathbb{C}$, $t \to t$ $\qquad \gamma_2 : [0,1] \to \mathbb{C}$, $t \to 1+t$

$\gamma_3 : [0,1] \to \mathbb{C}$, $t \to i + (1-t)$ $\qquad \gamma_4 : [0,1] \to \mathbb{C}$, $t \to (1-t)\,i$

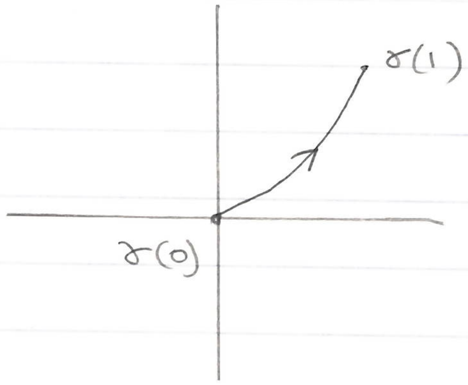
(3) A circle with center at z_0, and radius r has a parametrization

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$$
$$t \longmapsto z_0 + re^{it}$$



smooth, closed, simple

(4) $\gamma : [0,1] \longrightarrow \mathbb{C}$
$t \longrightarrow t + it^2$



smooth, non closed, simple

To define the complex line integrals recall that continuous function of a real valued function g on an interval $[a, b]$ is Riemann integrable, ie $\int_a^b g(t)\,dt$ exists

For a complex valued function $g : [a, b] \to \mathbb{C}$ we can define the integral

$$\int_a^b g(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

where $g(t) = u(t) + i\,v(t)$

Defn Suppose $\gamma : [a,b] \to \mathbb{C}$ is a smooth path, and $f : \Omega \to \mathbb{C}$ is a complex valued function which is defined and continuous on γ we define the integral of f along γ

by
$$\boxed{\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt}$$

Since $g(t) = [f(\gamma(t))]\,\gamma'(t) : [a,b] \to \mathbb{C}$ is continuous on $[a,b]$, the integral on the right is meaningful, as long as we show that it is independent of the parametrization of γ.

let $\tilde{\gamma} : [c,d] \to \mathbb{C}$ be another parametrization s.t

$\tilde{\gamma}(s) = (\gamma \circ \sigma)(s)$ for some $\sigma : [c,d] \to [a,b]$
with $\sigma \in C^1$, $\sigma'(s) > 0$

Then
$$\int_{\tilde{\gamma}} f(z)\,dz = \int_c^d f(\tilde{\gamma}(s))\,\tilde{\gamma}'(s)\,ds$$

$$= \int_c^d f(\gamma(\sigma(s)))\,\gamma'(\sigma(s)) . \sigma'(s)\,ds$$

letting $t = \sigma(s)$, $dt = \sigma'(s)\,ds$ gives $= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_\gamma f(z)\,dz$

The following properties of path integrals follow easily from the properties of the Riemann integral.

Prop 3-1 Let $f, g : \Omega \to \mathbb{C}$ continuous γ, γ_1, γ_2 are piecewise smooth curves in Ω. $a, b \in \mathbb{C}$. Then

① $$\int_\gamma (af + bg)(z)\,dz = a\int_\gamma f(z)\,dz + b\int_\gamma g(z)\,dz$$

② If γ^- is the curve γ with reverse orientation then

$$\int_{\gamma^-} f(z)\,dz = -\int_\gamma f(z)\,dz$$

③ $$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

④ $$\left|\int_\gamma f(z)\,dz\right| \leq \sup_{z\in\gamma} |f(z)|\ \mathrm{length}(\gamma)$$

$$\sup_{z\in\gamma} |f(z)| = \sup_{t\in[0,b]} |f(\gamma(t))|$$

$$\mathrm{length}(\gamma) = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} |\gamma'(t)|\,dt$$

Proof ① Follows from the linearity of the Riemann integral

② If $\gamma : [a,b] \to \mathbb{C}$, then $\gamma^- : [a,b] \to \mathbb{C}$
$t \to \gamma(a+b-t)$

$$(\gamma^-)'(t) = -\gamma'(a+b-t)$$

$$\int_{-\gamma} f(z)dz = -\int_a^b f(\gamma(a+b-t))\,\gamma'(a+b-t)\,dt$$

$u = b+a-t$
$du = -dt$

$$= \int_b^a f(\gamma(u))\,\gamma'(u)\,du = -\int_a^b f(\gamma(u))\,\gamma'(u)\,du$$

$$= -\int_\gamma f\,dz$$

③ Exercise

④ $$\left| \int_\gamma f(z)dz \right| = \left| \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\,\gamma'(t)\,dt \right|$$

$$\leq \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\leq \sup_{t\in[a,b]} |f(\gamma(t))| \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} |\gamma'(t)|\,dt$$

$\blacksquare$