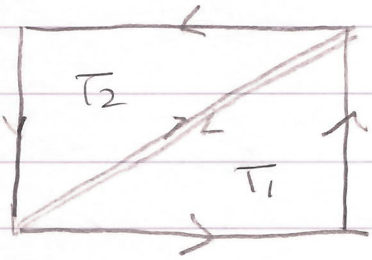
This follows immediately from the theorem and by dividing the rectangle into 2 Δ's



$$\int_R f\,dz = \int_{T_1} f\,dz + \int_{T_2} f\,dz$$
$$= 0$$

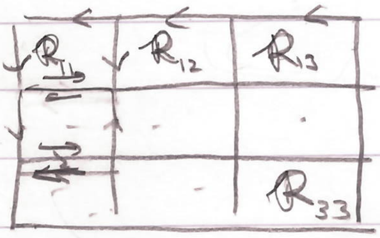
For future results, for example for the derivation of Cauchy's integral formula a minor extention of this result is useful.

Thm 11 (Goursat) If a function f is continuous in an open set Ω and analytic in $\Omega\setminus\{z_0\}$ for some $z_0 \in \Omega$, then

$$\int_R f(z)\,dz = 0 \quad \text{for every closed rectangle } \mathcal{R} \subset \Omega$$
$$(\partial\mathcal{R} = R)$$

Proof. Fix a closed triangle $\mathcal{R} \subset \Omega$.
We assume $z_0 \in \mathcal{R}$ otherwise the conclusion follows from the first version above.

Given a positive integer n we subdivide R into n^2 congruent rectangles, $\partial R = R$.



Once again it follows that

$$\int_R f\,dz = \sum_{k=1}^{n} \sum_{l=1}^{n} \int_{R_{kl}} f\,dz$$

If $z_0 \notin R_{kl}$ then $\displaystyle\int_{R_{kl}} f(z)\,dz = 0$ by the first version

If $z_0 \in R_{kl}$ then

$$\left| \int_{R_{kl}} f(z)\,dz \right| \leq M \operatorname{perimeter}(R_{kl}) = \frac{ML}{n}$$

where $M = \max_{z \in R} |f|$ is the maximum of the continuous function $|f|$ on compact set R.

The point z_0 cannot belong to more than 4 subrectangles.

Hence

$$\left| \int_R f(z)\,dz \right| = \left| \sum_{z_0 \in R_{kl}} \int_{\partial R_{kl} = R_{kl}} f(z)\,dz \right|$$

$$\leq \sum_{z_0 \in R_{kl}} \left| \int_{R_{kl}} f\,dz \right| \leq 4\frac{ML}{n} \to 0 \text{ as } n \to \infty$$

$\square$

To prove the Cauchy's thm in a disc we will need the local existence of primitives

We have the following theorem

Theorem 2.1 A holomorphic function in an open disc D has a primitive in that disc.

Rk. We will prove the following version which assumes that f is continuous in D and that its integral along rectangles whose sides parallel to the coordinate axes vanish. which then we'll use to give a slightly stronger form of Cauchy's theorem.

Thm 2-1' Let D be an open disc in $\mathbb{C}$ and f be a continuous function in D with the property that

$$\int_R f\,dz = 0$$ for every closed rectangle $\mathcal{R}$ with $\partial\mathcal{R} = R$ in D whose sides are parallel to the coordinate axis. Then f has a primitive in D.

Before we prove Thm 2.1', note that we have as a corollary

Thm 2.2' (Cauchy's theorem for a disc)
Suppose D is an open disc in $\mathbb{C}$
f a function holomorphic in D
(or more generally continuous in D, and holomorphic in $D \setminus \{z_0\}$ for some $z_0 \in D$)
Then
$$\int_\gamma f\,dz = 0 \quad \text{for every closed,}$$
piecewise smooth path in D.

Proof of Cauchy's thm Suppose f is cont. in D and holom $\forall\, z \in D \setminus \{z_0\}$
Then by Goursat's thm Thm' ~~1~~-1
$$\int_R f(z)\,dz = 0 \quad \text{for every closed rectangle } R \subset D$$
with $\partial R = R$.
(including the ones whose sides are parallel to axes)
By Thm 2.1' f has a primitive in D

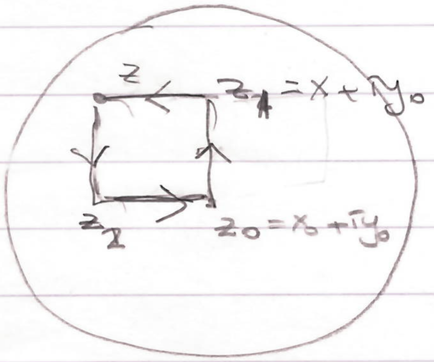
By Thm ~~3~~.2 of Chapter 1, Cor 3.3 $\int_\gamma f\,dz = 0$
for every piecewise smooth path γ in D.

Let f be continuous on a disc D.

Proof of Thm 2-1'. Let $z_0 = x_0 + iy_0$ be the center of the disc D.

For an arbitrary point $z \in D$, $z \neq z_0$, $z = x + iy$

let $z_1 = x + iy_0$
$z_2 = x_0 + iy$



By assumption

$$(*) \quad \int_{z_0}^{z_1} f(w)\,dw + \int_{z_1}^{z} f(w)\,dw + \int_{z}^{z_2} f\,dw + \int_{z_2}^{z_0} f\,dw = 0$$

This sum represents either $\int_{\Gamma} f(w)\,dw$ or $-\int_{\Gamma} f(w)\,dw$

depending on the location of z.

We define $F : D \longrightarrow \mathbb{C}$ as follows

For $z \in D$,

$$F(z) := \int_{z_0}^{z_2} f(w)\,dw + \int_{z_2}^{z} f(w)\,dw \,. \quad (A)$$

which is by $(*)$

$$= \int_{z_0}^{z_1} f(w)\,dw + \int_{z_1}^{z} f(w)\,dw \,. \quad (B)$$

Parametrizing the line segments we have

$$F(z) = \underbrace{i\int_{y_0}^{y} f(x_0 + it)\,dt}_{\text{indep of } x} + \int_{x_0}^{x} f(t+iy)\,dt \qquad (A)$$

and

$$F(z) = \underbrace{\int_{x_0}^{x} f(t+iy_0)\,dt}_{\text{indep of } y} + i\int_{y_0}^{y} f(x+it)\,dt \qquad (B)$$

Using (A) and Fund. thm of Analysis: have

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x) \quad \text{if } g:(a-r,a+r)\to\mathbb{C} \text{ is continuous}$$

with $g(t) = f(t+iy)$

$$F_x(z) = f(x+iy) = f(z)$$

Similarly using B and again fund $\frac{d}{dy}\int_a^y h(t)\,dt = h(y)$

$h(t) = f(x+it)$

we get

$$F_y(z) = if(x+iy) = if(z)$$

(For both parts we used that the first integral in (A) and (B) are indep of x, y resp.)

Hence it follows that F_x, F_y exist and continuous ie $F \in C^1(D)$

Since $F_x(z) = f(z)$, $F_y(z) = i f(z)$
$(f(z) = -i F_y(z))$

if we write $F(z) = u + iv$ then this gives $f(z) = F_x(z) = u_x + i v_x$
$= -i F_y(z) = -i(u_y + i v_y) = v_y - i u_y$

Hence $u_x = v_y$ and $v_x = -u_y$

Hence $F \in C^1$ and F satisfies CR eqns
By Thm 2.4 F is holomorphic in Ω
and $F'(z) = \dfrac{\partial F}{\partial x}(z) = f(z)$

ie F is a primitive of f $\blacksquare$

We can use Cauchy Thm for a disc to calculate some integrals.

<u>Example.</u> We can show by parametrizing the circle that

$$\int_{|w-z_0|=r} \frac{1}{z-z_0}\,dz = 2\pi i \quad \text{for every } r>0.$$

Indeed the circle of center z_0 and radius r has parametrization $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

$$\int_{|w-z_0|=r} \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{re^{it}}\, i r e^{it}\,dt = i\,2\pi.$$