Since $F_x(z) = f(z)$, $F_y(z) = if(z)$
$(f(z) = -i F_y(z))$

if we write $F(z) = u + iv$ then this gives $f(z) = F_x(z) = u_x + iv_x$
$= -i F_y(z) = -i(u_y + iv_y) = v_y - iu_y$

Hence $u_x = v_y$ and $v_x = -u_y$

Hence $F \in C^1$ and F satisfies CR eqns
By Thm 2.4 F is holomorphic in Ω
and $F'(z) = \frac{\partial F}{\partial x}(z) = f(z)$

ie F is a primitive of f $\blacksquare$

We can use Cauchy Thm for a disc to calculate some integrals.

Example. We can show by parametrizing the circle that

$$\int_{|w-z_0|=r} \frac{1}{z-z_0}\,dz = 2\pi i \quad \text{for every } r>0.$$

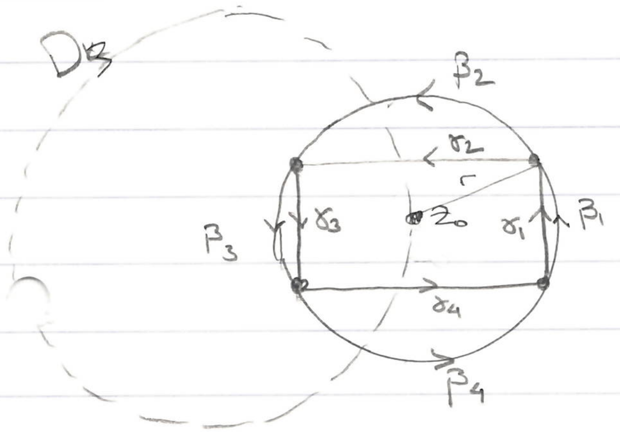
Indeed the circle of center z_0 and radius r has parametrization $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

$$\int_{|w-z_0|=r} \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{re^{it}}\, ire^{it}\,dt = i\,2\pi.$$

Now
Using Cauchy's thm we can also show that

$$\int_R \frac{1}{z - z_0}\, dz = 2\pi i$$ for <u>any</u> rectangle R with center at z_0

Note $\int_R \frac{1}{z - z_0}\, dz$ is <u>not</u> zero since $\frac{1}{z - z_0}$ is not cont. at z_0 and hence Cauchy thm does not apply directly.
We can use it though as follows:



let $C_r(z_0)$ be the circle that circumscribes the rectangle R.

$$R = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$
$$C_r(z_0) = \beta_1 + \beta_2 + \beta_3 + \beta_4.$$

For each $1 \le k \le 4$ we choose an open disc D_k so that the trajectory of the closed path $\gamma_k - \beta_k$ is in D_k and $f(z) = \frac{1}{z - z_0}$ is holom in D_k

Now we apply Cauchy's thm to $\frac{1}{z - z_0}$ in disc D_k and $\gamma_k - \beta_k \in D_k$ to get

$$\int_{\gamma_k - \beta_k} f(z)\, dz = 0 \Rightarrow \int_{\gamma_k} \frac{1}{z - z_0}\, dz = \int_{\beta_k} \frac{1}{z - z_0}\, dz$$

But then $$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} \frac{1}{z-z_0}\,dz = \int_{\beta_1+\beta_2+\beta_3+\beta_4} \frac{1}{z-z_0}\,dz$$

and hence $$\int_{R} \frac{1}{z-z_0}\,dz = \int_{C_r(z_0)} \frac{1}{z-z_0}\,dz = 2\pi i$$ ■

We next prove Cauchy's integral formula from which we will deduce many properties of holomorphic functions.

Cauchy's integral Formula

Thm (CIF) (Thm 4.1, chap 2)
Suppose f is holom in an open set Ω that contains the closure of a disc D. If C denotes the boundary circle of the disc with positive orientation (ie ccw), then
$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\,d\xi \quad \text{for any } z \in D$$

Rk Note CIF says values of f on D is determined by its boundary values on the circle

Proof let $z \in D_r(z_0)$. $\overline{D_r(z_0)} \subset \Omega$. $\exists\ \varepsilon > 0$ s.t. $D_{r+\varepsilon}(z_0) \subset \Omega$. For $w \in D_{r+\varepsilon}(z_0)$ we define

$$g(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & w \neq z \\ f'(z) & w = z \end{cases}$$

Then $g : D_{r+\varepsilon}(z_0) \longrightarrow \mathbb{C}$ is continuous

and away from z is holomorphic

By Cauchy's thm, Thm 2-2' applied to g we have

$$\int_{C_r(z_0)} g(w)\, dw = 0$$

ie $$\int_{C_r(z_0)} \frac{f(w) - f(z)}{w - z}\, dw = 0$$ (Note on $C_r(z_0)$ $w \neq z$ since $z \in D_r(z_0)$)

Hence $$\int_{C_r(z_0)} \frac{f(w)}{w - z}\, dw = f(z) \int_{C_r(z_0)} \frac{dw}{w - z}$$

To finish we claim $$\int_{C_r(z_0)} \frac{dw}{w - z} = 2\pi i$$

Claim $\displaystyle\int_{C_r(z_0)} \frac{dw}{w-z} = 2\pi i.$

Rmk This was already in one of the exercises. Here we give another proof

Proof $C_r(z_0)$ has parametrization : $\gamma(t) = z_0 + re^{it}$ $0 \le t \le 2\pi$
$\gamma : [0, 2\pi] \to \mathbb{C}$,

It also has the following parametrization

$$\tilde{\gamma}(s) = z + \rho(s) e^{is}$$

where $\rho : [0, 2\pi] \to \mathbb{R}$ $\rho(s) := |\gamma(t(s)) - z|$

clearly ρ is smooth

$$\tilde{\gamma}'(s) = \rho'(s) e^{is} + i\rho(s) e^{is}$$

$$\int_{C_r(z_0)} \frac{dw}{w-z} = \int_0^{2\pi} \frac{\rho'(s) e^{is} + i\rho(s) e^{is}}{\rho(s) e^{is}} ds$$

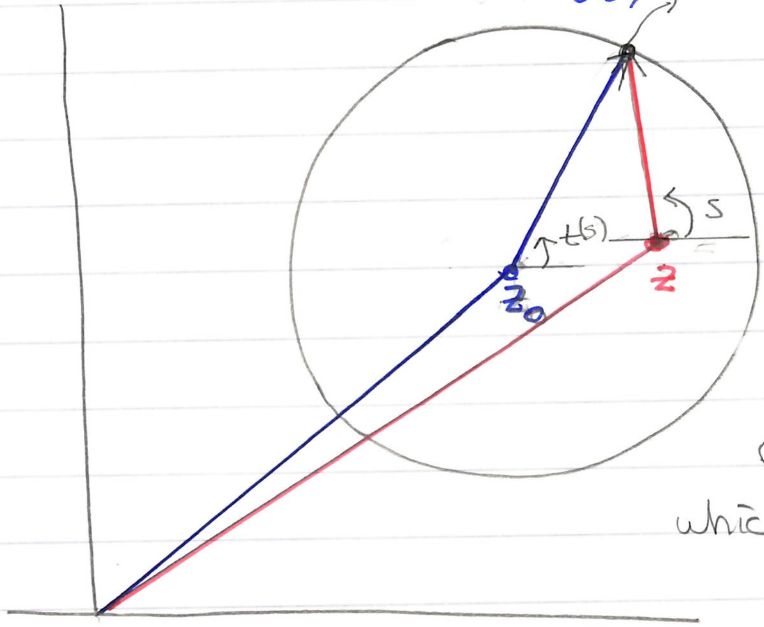
$$= \underbrace{\int_0^{2\pi} \frac{\rho'(s)}{\rho(s)} ds}_{\ln|\rho(s)|\Big|_{s=0}^{s=2\pi}} + i \underbrace{\int_0^{2\pi} ds}_{2\pi i}$$

real integral.

$= 0$ since $\rho(2\pi) = \rho(0)$

$\gamma(t) = z_0 + re^{it} = z + \rho e^{is}$

$= \tilde{\gamma}(s) = \gamma(t(s))$



Here t changes with s

$\rho = \rho(s) = |\gamma(t(s)) - z|$

which we write as

$= |\gamma(s) - z|$

by abuse of notation

$\tilde{\gamma}(s) = z + \rho(s)e^{is}$ is the new parametrization

$\sigma : [0, 2\pi] \longrightarrow [0, 2\pi]$ is the change of variables

$s \longrightarrow t(s)$

$\tilde{\gamma} = \gamma \circ \sigma$

Before we give important theoretical applications of Cauchy's theorem and Cauchy integral formula we'll look at one more example of contour shifting which helps us to evaluate certain integrals.

Example We'll show that $e^{-\pi x^2}$ is its own Fourier transform:

For a function $f : \mathbb{R} \to \mathbb{C}$ which is Riemann integrable on every $[a, b]$ and $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges, its Fourier transform

$$\boxed{\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx}$$

is well defined for all $\xi \in \mathbb{R}$

we want to show that if $f(x) = e^{-\pi x^2}$ then

$$\hat{f}(\xi) = e^{-\pi \xi^2}$$

ie

w.t.s.

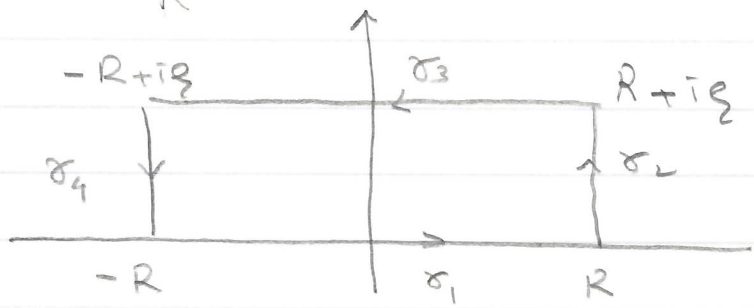
$$e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx$$

If $\xi = 0$ this gives $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$

which we know from Analysis

we first suppose $\xi > 0$ and let

$f(z) = e^{-\pi z^2}$ then $f(z)$ is entire and in particular holomorphic in the piecewise smooth contour $\gamma_R = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$



Hence using Cauchy's thm

$$\int_{\gamma_R} f(z)\,dz = 0$$

Note on γ_1 : $$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} e^{-\pi x^2}\,dx$$

on γ_3 $$\int_{\gamma_3} f(z)\,dz = \int_{R}^{-R} e^{-\pi(x+i\xi)^2}\,dx$$

$$= -\int_{-R}^{R} e^{-\pi(x^2 + 2\pi i x\xi)} \cdot e^{+\pi\xi^2}\,dx$$

$$= -e^{+\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} \cdot e^{-2\pi i x\xi}\,dx$$

As $R \to \infty$ the first integral over $\gamma_1 = 1$

the integral over γ_2 gies

$$e^{-\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi i x\xi}\,dx$$

On the vertical side on the right

$$\int_{\gamma_2} f(z)\,dz = \int_0^{\xi} f(R+iy)\, i\,dy$$

$$= \int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)}\, i\,dy$$

For fixed ξ, the integral can be bounded with

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \xi \sup_{0\le y\le \xi} \left| e^{-\pi R^2} \cdot e^{-\pi i R y} \cdot e^{\pi y^2}\right|$$

$$\le C e^{-\pi R^2}$$

A similar bound holds for the γ_4

Hence as $R\to\infty$ both integrals go to 0.

And we obtain that $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = 0$

$$= \lim_{R\to\infty}\left[\int_{\gamma_1} + \dots \int_{\gamma_4} f(z)\,dz\right]$$

$$= 1 + 0 - e^{\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x}\,dx + 0$$

$$\Longrightarrow \boxed{e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2}\cdot e^{2\pi i x\xi}}$$

∎

Next we see that Cauchy's theorem and CIF will imply fundamental properties of holomorphic functions.

Namely we'll see that they're enough to prove

1. If $\Omega \subset \mathbb{C}$ open and $f \in \mathcal{H}(\Omega)$ then $f' \in \mathcal{H}(\Omega)$. Hence f is only differentiable
   And if $z_0 \in \Omega$ and $r > 0$ s.t $D_r(z_0) \subset \Omega$ then f has a power series expansion at z_0
   $$f(z) = \sum a_n (z - z_0)^n \quad \forall\ z \in D_r(z_0)$$
   ie f is analytic in $D_r(z_0)$

2. If f is entire, ie $f : \mathbb{C} \to \mathbb{C}$ holom everywhere then f is constant

3. Fund. thm of algebra holds
   ie any polynomial $p(z) \in \mathbb{C}[z]$ of degree n, has n roots in $\mathbb{C}$ (counted w/ multiplicity).

4. If f, g are holom in Ω and $f(z) = g(z)$ $\forall z$ in some sequence of distinct points with a limit point in Ω then $f(z) = g(z)$ $\forall z \in \Omega$. in particular if f, g

agree on an open set of Ω, they agree on all Ω.

We start with the following theorem

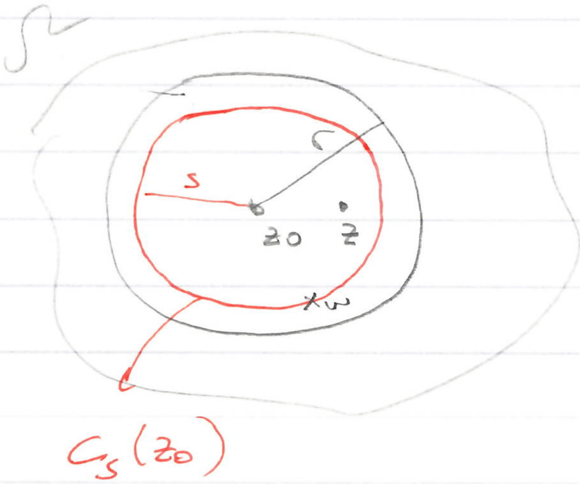
Thm 4.4 Suppose f is holomorphic in an open set Ω. Let $z_0 \in \Omega$, $r > 0$ so that $\overline{D_r(z_0)} \subset \Omega$. Then f has a power series expansion at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } z \in D_r(z_0)$$

and the coefficients a_n are given by the formula

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \forall n \geq 0.$$

Proof. Let z_0, r be as above so that $\overline{D_r(z_0)} \subset \Omega$. Fix $s \in (0, r)$ and let $C_s(z_0)$ be the circle of radius s with center z_0. Then $\gamma = C_s(z_0) \subset \Omega$



and by CIF

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z}\, dw$$

$\forall z \in D_s(z_0)$

The trick is to write

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)}$$

$$= \frac{1}{(w-z_0)}\left(\frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)}\right)$$

Since we are integrating on γ, for $w \in \gamma$

$$\left|\frac{z-z_0}{w-z_0}\right| = \frac{|z-z_0|}{s} < 1 \quad (z \in D_s(z_0))$$

Hence $$\frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n \quad \text{for } w \in \gamma,\ z \in D_s(z_0)$$

Convergence is uniform since the bound $\frac{|z-z_0|}{s}$ for $\left(\frac{z-z_0}{w-z_0}\right)$ is indep of $w \in \gamma$. Hence we can interchange the series and the integral.

Hence we obtain

$$f(z) = \frac{1}{2\pi i}\int_\gamma f(w)\sum_{n=0}^{\infty}(z-z_0)^n(w-z_0)^{-n-1}\,dw$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw.$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

where $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{n+1}} \, dw$.

Hence we have that in $D_s(z_0)$, f is the sum of the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

We've seen that power series in their disc of convergence are differentiable with derivatives given by termwise differentiation (Thm 2.6)

Hence in $D_s(z_0)$ we have

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} (z - z_0)^n \quad \forall z \in D_s(z_0)$$

Being a power series $f'(z)$ is also holomorphic in $D_s(z_0)$. Inductively we get f is differentiable ∞ly often for $z \in D_s(z_0)$ and evaluating at $z = z_0$

gives $a_0 = f(z_0)$

$1 a_1 = f'(z_0)$

$\vdots$

$n! \, a_n = f^{(n)}(z_0)$

Hence $a_n = \frac{f^{(n)}(z_0)}{n!}$ is independent of s.