11-10-23

and $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ conv. for all

$z \in D_s(z_0)$ $\forall s \in (0,r)$. Hence the radius of convergence of

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ is at least } r$$

and we are done. ☒

Note that the proof also gives

$$a_n = \frac{1}{2\pi i} \int_{C_s(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} dw = \frac{f^{(n)}(z_0)}{n!}$$

In fact we have

Cor 4.2 If $f \in \mathcal{H}(\Omega)$, f is ∞ly differentiable in Ω.
Moreover if $z_0 \in \Omega$ s.t $r > 0$ $\overline{D_r(z_0)} \subset \Omega$ then for any $z \in D_r(z_0)$

$$\boxed{f^{n}(z) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}} dw}$$

Cauchy Int. For. for derivatives

Proof The fact that f' is differentiable follows because $\forall z_0 \in \Omega$, $\exists r > 0$ s.t $D_r(z_0) \subset \Omega$. By Thm 4.4 f has a power series expansion there, hence holom in $D_s(z_0)$ for some $s<r$. Since power series are ∞ly diff in their disc of convergence we have, f is diff ∞ly'ly often at z_0. Since z_0 was arbitrary f is ∞ly diff in Ω.

To prove the IF. Note $n=0$ is simply the CIF in Thm 4.1. We'll use induction

suppose

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z)^n}\,dw$$

for any $z \in D_r(z_0)$

Then for h small, so that $z+h$, z are away from $C_s(z_0)$

$$\frac{f^{n-1}(z+h)-f^{n-1}(z)}{h} = \frac{(n-1)!}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{h}\left[\frac{1}{(w-z-h)^n}-\frac{1}{(w-z)^n}\right] dw.$$

use $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\dots+b^{n-1})$

w/ $a=\frac{1}{w-z-h}$ $\quad b=\frac{1}{w-z}$ and take the limit as $h\to 0$

Note $(a-b)\to\frac{1}{(w-z)^2}$ and $(a^{n-1}+\dots+b^{n-1})\to\frac{n}{(w-z)^{n-1}}$ as $h\to 0$

to get $\lim \frac{f^{n-1}(z+h)-f^{n-1}(z)}{h}$

$$= \frac{n-1!}{2\pi i}\int_{C_s(z_0)} f(w)\left[\frac{1}{(w-z)^2}\cdot\frac{n}{(w-z)^{n-1}}\right]dw$$

$$= \frac{n!}{2\pi i}\int_{C_s(z_0)} \frac{f(w)}{(w-z)^{n+1}}\,dw$$

$\boxtimes$.

Remark Thm 4-4 says that a holomorphic function f can be locally developed as a power series around each point of the definition domain Ω.

Explicitly for each $z_0 \in \Omega$, $\exists$ $D_r(z_0) \subset \Omega$

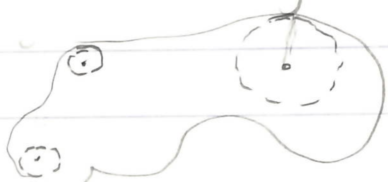
and a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

which converges $\forall z \in D_r(z_0)$ and represents the function f in $D_r(z_0)$

Due to this power series development we have that holomorphic functions are exactly the functions which are everywhere representable as a power series (with a positive radius of convergence (Recall every power series represents a holom. func in their disc of convergence.

And this is why we have the words "holomorphic", "analytic" used interchangeably in various sources.

Note that the power series $\sum a_n (z-z_0)^n$ might not represent $f(z)$ in all of Ω, but represents it at least in a disc whose radius is the distance from the point to the boundary of Ω.



Cor 4.3 (Cauchy inequalities) With the assumptions as in Cor 4.2 we have

$$|f^{(n)}(z_0)| \leq \frac{n! \sup_{|w-z_0|=r} |f(w)|}{r^n} = \frac{n! \, \|f\|_{C_r(z_0)}}{r^n}$$

where $\|f\|_{C_r(z_0)}$ is the sup of f on $C_r(z_0)$.

Proof This follows from

$$|a_n| = \left|\frac{f^n(z_0)}{n!}\right| = \left|\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} \, dw\right|$$

$$|n! \, a_n| = |f^n(z_0)| = \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} \, ire^{i\theta} d\theta \right|$$

$$\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + re^{i\theta})|}{r^n} \, d\theta$$

$$\leq \frac{n!}{2\pi} \frac{\|f\|_c}{r^n}$$

An immediate corollary of these results is the remarkable

Liouville's thm

Thm (Liouville's theorem) (Cor 4.5).
If $f \in \mathcal{H}(\mathbb{C})$ and if f is bounded then f is constant.

Proof Since $\mathbb{C}$ is connected, it is enough to show $f' = 0$ (Cor. 3.4. Chap. 1)

let $z_0 \in \mathbb{C}$, $r > 0$ then $D_r(z_0) \subset \mathbb{C}$ $\forall r$
and since f is holom on all $\mathbb{C}$, we have using Cauch inequalities

$$|f'(z_0)| \leq \frac{\|f\|_{C_r(z_0)}}{r}$$

By assumption f is bounded; $\exists$ $M \geq 0$ s.t
$\forall z \in \mathbb{C}$ $|f| < M$

Hence $|f'(z_0)| < \frac{M}{r}$ $\forall r$

letting $r \to \infty$ we get $f'(z_0) = 0$
since z_0 was arbitrary $f'(z) = 0$ $\forall z \in \mathbb{C}$
and hence f is constant

Cor (Fundamental Thm of algebra)
Every poly $P(z) = a_n z^n + \dots + a_0$ of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If the roots are $w_1, \dots w_n$ (with possible repetitions) then

$$P(z) = a_n (z - w_1) \dots (z - w_n).$$

Proof. We first show that $p(z)$ has a root in $\mathbb{C}$.

Suppose not then the function $Q(z) = \dfrac{1}{p(z)}$ is in $\mathcal{H}(\mathbb{C})$

If $Q(z)$ is bounded then it would be a constant by Liouville's thm. which then contradicts that $P(z)$ is not constant, i.e $\deg P \geq 1$ with $P(z) = a_n z^n + \dots + a_0$ where $a_n \neq 0$, $a_i \in \mathbb{C}$.

<u>Claim</u> $Q(z)$ is bounded. For $z \neq 0$ we have

$$|P(z)| = |a_n z^n + \dots + a_0|$$

$$\geq |a_n||z|^n - \sum_{i=0}^{n-1} |a_i||z|^i$$

$$\geq |z|^n \left(|a_n| - \frac{|a_0|}{|z|^n} - \dots - \frac{|a_{n-1}|}{|z|} \right)$$

Hence $|P(z)| \to \infty$ as $|z| \to \infty$

Hence $|Q(z)| \to 0$ as $|z| \to \infty$

Hence $\exists\, r > 0$ s.t $|Q(z)| \leq 1$ when $|z| \geq r$ ($Q(z)$ is a continuous function)

but $Q(z)$ is continuous hence bounded on the compact set $|z| \leq r$
say $|Q(z)| < m$ for some $m \in \mathbb{R}$

Choose $M = \max\{m, 1\}$ then

$$|Q(z)| \leq M \quad \forall z \in \mathbb{C}$$

Hence $Q(z)$ is constant by Liouville's thm.

Hence P has a root, say $w_1 \in \mathbb{C}$. Then writing $z = (z - w_1) + w_1$ we have

$$P(z) = a_n\left((z - w_1) + w_1\right)^n + \cdots + a_0$$

$$= b_n (z - w_1)^n + \cdots + b_1 (z - w_1) + b_0$$

with new coefs $b_1, \ldots b_{n-1}$
$b_n = a_n$

using $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^n b^{n-k}$

Since $P(w_1) = 0$, we get $b_0 = 0$

Hence $P(z) = (z - w_1)\left[b_n (z - w_1)^{n-1} + \cdots + b_1\right]$

$$= (z - w_1)\, \tilde{P}(z)$$

where $\tilde{P}(z)$ is a poly of degree $n-1$
By induction on the degree of the poly we get the result ☒