17-10-23

(76)

Next we discuss the principle of analytic continuation (of identities) which says that: if Ω is open and connected, $f \in \mathcal{H}(\Omega)$ and f vanishes on an infinite set Z of distinct points with a limit point $z_0 \in \Omega \setminus Z$ then $f \equiv 0$.

Remark (1) Holomorphic functions can have ∞ly many zeroes

e.g $f(z) = \cos z$ (or $\sin z$)

has zeroes for $z = (2k+1)\frac{\pi}{2}$ (or $z = \pi k$)

But we'll see that the zeroes are isolated, i.e for each zero z_0 of f $\exists$ a neighbourhood of z_0 with no other zeroes.

(2) There are holomorphic functions with no zeroes, eg constant function, e^{z}.

We start by the defn of a limit point

Defn $z_0 \in \mathbb{C}$ is a limit point of a set Ω if $\exists$ a sequence $(z_n)_{n\geq 1}$ in $\Omega \setminus \{z_0\}$ (i.e $z_n \neq z_0$) s.t $\lim z_n = z_0$.

Hence $\forall \varepsilon > 0$, $\Omega \cap (D_\varepsilon(z_0) \setminus \{z_0\}) \neq \emptyset$.

Rmk If $\Omega = [-1, 1] \cup \{2i\}$ then the $z_n \neq z_0$ condition avoids the case $2i$ is a limit point of Ω. Since otherwise we could take $z_n = 2i$ $\forall n$.

We next define order of zero of f at z_0.

Defn Ω open, $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$.
The order of zero of f at z_0, (or order of vanishing at z_0)
denoted by $\operatorname{ord}_{z_0}(f)$ or $n_{z_0}(f)$ or $\nu_{z_0}(f)$

is either ∞ if $f^{(k)}(z_0) = 0$ $\forall k \geq 0$
or it is the smallest integer k
s.t $f(z_0) = f'(z_0) = \dots = f^{(k-1)}(z_0) = 0$
$f^{(k)}(z_0) \neq 0$.

If $f(z_0) \neq 0$ then $k = 0$.

$$\operatorname{ord}_{z_0}(f) = \min\{k \geq 0 \mid f^{(k)}(z_0) \neq 0\}$$

We have the following

Proposition Let Ω be open, $f \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$
Then (1) if $\operatorname{ord}_{z_0} f = \infty$ then $f(z) = 0$

for any $z \in D_r(z_0)$ s.t $D_r(z_0) \subset \Omega$.
ie f is locally zero

2 If $\text{ord}_{z_0}(f) \neq \infty$ then $\exists!\, h \in \mathcal{H}(D_r(z_0))$

and $n \in \mathbb{Z}$, $n \geq 0$ s.t

$$f(z) = (z - z_0)^n h(z) \quad \forall z \in D_r(z_0)$$

where $h(z_0) \neq 0$, $n = \text{ord}_{z_0}(f)$

3 For any $f, g \in \mathcal{H}(\Omega)$ we have

$$\text{ord}_{z_0}(f+g) \geq \min(\text{ord}_{z_0} f, \text{ord}_{z_0} g)$$

$$\text{ord}_{z_0}(fg) = \text{ord}_{z_0}(f) + \text{ord}_{z_0}(g)$$

Proof 1 f is holomorphic on Ω, Ω open

Hence by Thm 4.4, $\exists r > 0$ s.t

$\forall z \in D_r(z_0) \subset \Omega$ we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Since $\text{ord}_{z_0}(f) = \infty$, $f^{(n)}(z_0) = 0 \quad \forall n$

Hence $f(z) = 0 \quad \forall z \in D_r(z_0) \subset \Omega$.

2. If $\mathrm{ord}_{z_0}(f) \neq \infty$, then by defn

$\exists\ k \geq 0$ s.t $f(z_0) = \dots = f^{(k-1)}(z_0) = 0$

and $f^{(k)}(z_0) \neq 0$

Again using Thm 4-4, $\exists\, r > 0$ s.t $D_r(z_0) \subset \Omega$

and $\forall z \in D_r(z_0)$ we have the power series repn

$$f(z) = \frac{f^{k}(z_0)}{k!}(z - z_0)^k + \sum_{n=k+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$= (z - z_0)^k \left[\frac{f^{k}(z_0)}{k!} + \sum_{m=1}^{\infty} \frac{f^{m+k}(z_0)}{(m+k)!}(z - z_0)^m \right]$$

$$= (z - z_0)^k \left[\sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m \right]$$

Hence if we define

$$h(z) := \sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m \quad \forall z \in D_r(z_0)$$

Then $h(z) \in \mathcal{H}(D_r(z_0))$ since it is given by a convergent power series

and $h(z_0) = \frac{f^{(k)}(z_0)}{k!} \neq 0$.

Note since $h \in \mathcal{H}(D_r(z_0))$ it is also continuous there and since $h(z_0) \neq 0$

$\exists\ 0 < \varepsilon < r$ s.t $h(z) \neq 0 \ \forall z \in D_\varepsilon(z_0)$.

Moreover h, n are unique since if

$$f(z) = (z - z_0)^n h(z) = (z - z_0)^m g(z)$$

with h, g holom. and $h(z_0) \neq 0$, $g(z_0) \neq 0$

Then if $m > n$ we get

$$f(z) = (z - z_0)^n (z - z_0)^{m-n} g(z)$$
$$= (z - z_0)^n h(z)$$

For $z \neq z_0$

$$h(z) = (z - z_0)^{m-n} g(z)$$

but now taking lim on both sides as $z \to z_0$ gives $h(z_0) = 0$ which is a contradiction unless $m = n$, and then $h(z) = g(z)$

(3). Note for any k

$$f^{(k)}(z_0) + g^{(k)}(z_0) = (f+g)^{(k)}(z_0)$$

Hence if $f^{(k)}(z_0) = 0 = g^{(k)}(z_0)$ then $(f+g)^{(k)}(z_0) = 0$

This imply that $\text{ord}_{z_0}(f+g) \geq \min(\text{ord}_{z_0} f, \text{ord}_{z_0} g)$

By part (2) we write $f(z) = (z - z_0)^{\text{ord}_{z_0} f} h_1(z)$

$$g(z) = (z - z_0)^{\text{ord}_{z_0} g} h_2(z)$$

with $\forall z \in D_r(z_0)$ $h_1(z_0) \neq 0$, $h_2(z_0) \neq 0$

then $$fg = (z - z_0)^{\text{ord}_{z_0} f + \text{ord}_{z_0} g} h_1(z) h_2(z)$$

with $(h_1 h_2)(z_0) \neq 0$

From this, using the power series expansion of fg or the uniqueness of n, h in part (2)

we get $\text{ord}_{z_0} f + \text{ord}_{z_0} g = \text{ord}_{z_0}(fg)$ ☒

As a corollory we get that the zeroes of a holom function are isolated. More precisely we have

Thm let $\Omega \subset \mathbb{C}$ open, $f \in \mathcal{H}(\Omega)$ $z_0 \in \Omega$. Assume $f(z_0) = 0$ ie $\operatorname{ord}_{z_0} f \geq 1$. If $\operatorname{ord}_{z_0} f \neq \infty$ then

$\exists\ \delta > 0$ s.t $f(z) \neq 0$ if $z \in D_\delta(z_0)$ and $z \neq z_0$

$\to$ f is not zero in $D_\delta(z_0)$ except at z_0.

Pf. We write
$$f(z) = (z - z_0)^n\, h(z)$$
with $n = \operatorname{ord}_{z_0} f$, $h(z_0) \neq 0$ $\forall\ z \in D_r(z_0)$.

let $z \neq z_0$, $z \in D_r(z_0)$. Then

$$f(z) = 0 \Leftrightarrow h(z) = 0.$$

But $h(z_0) \neq 0$ and $h(z)$ is continous on $D_r(z_0)$ so $\exists\ 0 < \delta \leq r$ s.t $h(z) \neq 0$ for $|z - z_0| < \delta$.

Hence $f(z) \neq 0$ $\forall\ z \in D_\delta(z_0)$ $\square$

Now we can state the principle of analytic continuation.

Thm (II. 4-8) Let $\Omega \subset \mathbb{C}$ open and connected

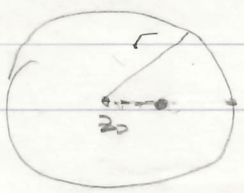
Let $f \in \mathcal{H}(\Omega)$. Let Z be an infinite set with a limit point $z_0 \in \Omega$, $z_0 \notin Z$. If $f(z) = 0$ $\forall z \in Z$, then $f = 0$.

Before we give the proof, we record the following immediate corollary

Corollary (4-9) Suppose f, g holom in Ω (open, connected) and $f(z) = g(z)$ for all z in some non-emty open sset $U \subset \Omega$ (or more generally for $z \in Z$, a sequence of distinct points with limit point in Ω) then $f(z) = g(z)$ $\forall z \in \Omega$.

Proof of cor: Apply Thm 4-8 to $f - g$ ▨

Note if $U \subset \Omega$ open, $U \neq \emptyset$, then $\exists$ $D_r(z_0) \subset U$ for some $z_0 \in U$, $r > 0$ and the sequence $\left\{ z_0 + \frac{r}{n+1} \right\}_{n=1}^{\infty} \subset D_r(z_0) \subset U$ has limit point $z_0 \in \Omega \setminus \left\{ z_0 + \frac{r}{n+1} \right\}_{n=1}^{\infty}$



Remark

1. The reason this result is called principle of analytic continuation is the following:

If $f \in \mathcal{H}(\Omega)$, Ω open connected and $\Omega \subset \tilde{\Omega}$ open connected then there is at most one $\tilde{f} \in \mathcal{H}(\tilde{\Omega})$ s.t $f(z) = \tilde{f}(z)$ $\forall z \in \Omega$. When such $\tilde{f}$ exists we say f has analytic continuation to $\tilde{\Omega}$.

(Note if $g(z) \in \mathcal{H}(\tilde{\Omega})$ is s.t $g(z) = f(z)$ $\forall z \in \Omega$

then $\tilde{f}(z) - g(z) = 0$ $\forall z \in \Omega$. Hence $\tilde{f}(z) = g(z)$ $\forall z \in \tilde{\Omega}$ by above theorem. Hence $\tilde{f}$ is unique).

2. The assumption that Ω is connected is essential. Since if $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_i \neq \emptyset$, $\Omega_1 \cap \Omega_2 = \emptyset$ then one can define $f, g : \Omega \to \mathbb{C}$ by $f|_{\Omega_1} = 1$ and $f|_{\Omega_2} = 0$ and $g \equiv 0$.

Then even though $f|_{\Omega_2} = g|_{\Omega_2}$ coincide f and g do not coincide in Ω.

(3) The condition that the limit point of zeroes is in Ω is also crucial

For example take $f : \mathbb{C}\setminus\{0\} =: \Omega \longrightarrow \mathbb{C}$

$$z \longmapsto \sin(\pi/z)$$

$$\sin\left(\frac{\pi}{z}\right) = \frac{e^{i\pi/z} - e^{-i\pi/z}}{2i}$$

$$f \in \mathcal{H}(\mathbb{C}\setminus\{0\}), \quad f \not\equiv 0$$

$$f(i) = \frac{e^{\pi} - e^{-\pi}}{2i} \neq 0.$$

$$f\left(\frac{1}{n}\right) = \sin(\pi n) = 0 \quad \forall n \geq 1$$

with $\lim \frac{1}{n} = 0$

But the limit point of zeroes is not in Ω

This example shows that the zeroes can converge to a boundary point,

Note we do have that the zeroes $\{\frac{1}{n}\}$ are isolated.

We'll prove the following thm which proves Thm 4.8.

Thm (4.8)' Let Ω be open connected $f \in \mathcal{H}(\Omega)$. Then the following are equivalent

ⓐ $f \equiv 0$

ⓑ $\exists$ a point $a \in \Omega$ s.t $f^{(n)}(a) = 0 \ \forall n \geq 0$

ⓒ $\{z \in \Omega \mid f(z) = 0\}$ has a limit point in Ω.

An immediate corollary of thm 4.8' is

Cor 4.9' (Identity thm) Let $f, g \in \mathcal{H}(\Omega)$ Ω open connected, $\Omega \neq \emptyset$

Then TFAE

ⓐ $f = g$

ⓑ $\exists$ a point $a \in \Omega$ s.t $f^{(n)}(a) = g^{(n)}(a)$ $\forall n \geq 0$

ⓒ $\{z \in \Omega \mid f(z) = g(z)\}$ has a limit point in Ω