The analytic continuation of the Riemann zeta function

We've seen that (Notes 12, 25.10.23)

(1) **Prop** (2.1, chapter 6) The series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ conv. abs. and unif on every half plane

$U_\delta := \{ s \in \mathbb{C} \mid \operatorname{Re} s \geq 1 + \delta \}$, $\delta > 0$ and

is holomorphic in $\{ s \in \mathbb{C} \mid \operatorname{Re} s > 1 \}$.

We've also seen

(2) **Prop** $\forall z \in \mathbb{H}$, $\theta(z) := \sum_{n=0}^{\infty} e^{\pi i n^2 z}$

converges and defines a holomorphic function in $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

From Notes 7, 10.10.23 we have

(3) For a function $f : \mathbb{R} \to \mathbb{C}$ which is Riemann integrable on every $[a, b]$ and $\int_{-\infty}^{\infty} |f(t)|\,dt$ converges, its Fourier transform is defined as

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x}\,dx$$

We also have shown that $f(x) = e^{-\pi x^2}$ has $\hat{f}(\xi) = e^{-\pi \xi^2}$

What we have not seen but can be proved is the following results about Fourier transform which can be found in Chapter 4 of the book.

For $a > 0$, denote by $\mathcal{F}_a$ the class of all functions f that satisfy the following 2 conditions:

① f is holomorphic in the horizontal strip $S_a = \{ z \in \mathbb{C} : |\operatorname{Im} z| < a \}$

② $\exists$ a constant $A > 0$ s.t
$$|f(x+iy)| \leq \frac{A}{1+x^2} \quad \forall x \in \mathbb{R}, |y| < a$$

(ie f is of moderate decay on each horizontal line $\operatorname{Im} z = y$ uniformly in $-a < y < a$)

<u>Example</u> $f(z) = e^{-\pi z^2} \in \mathcal{F}_a$ for all $a > 0$.

let $\mathcal{F} = \{ f \mid f \in \mathcal{F}_a$ for some $a \}$.

Fourier inversion says

Thm 2.2 if $f \in \mathcal{F}$, then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi \quad \forall x \in \mathbb{R}.$$

$$\left(\hat{\hat{f}}(x) = f(-x) \right)$$

Poisson summation says

Thm 2.4 if $f \in \mathcal{F}$ then

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

Cor Poisson summation formula applied to $f_t(x) = e^{-\pi t x^2}$, $t \in \mathbb{R}^+$

gives
$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \sum_{n=-\infty}^{\infty} t^{-1/2} e^{-\pi n^2/t}$$

Hence $\vartheta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi t n^2} = t^{-1/2} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} = \frac{1}{\sqrt{t}} \vartheta\left(\frac{1}{t}\right)$

$$\boxed{\vartheta(t) = \frac{1}{\sqrt{t}} \vartheta\left(\frac{1}{t}\right)}$$

Note $\vartheta(t) = \theta(it)$

We can now use this transformation of θ-function to give analytic continuation and func'l eqn of $\xi(s)$.

Thm 2.3 Let $\Lambda(s) := \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\xi(s)$, $\operatorname{Re} s > 1$

Then $\Lambda(s)$ has a meromorphic continuation to all of s-plane, with simple poles at $s=0, 1$, and satisfies the func'l equation

$$\Lambda(1-s) = \Lambda(s).$$

[Recall $\Gamma(s) := \int_0^\infty e^{-t}\, t^s \frac{dt}{t}$, $\operatorname{Re} s > 0$

Thm 1.3
Γ has analy. continuation to a meromorphic function on $\mathbb{C}$, with simple poles at $s = 0, -1, -2, \ldots$ and residue of $\Gamma(s)$ at $s=-n$ is $\frac{(-1)^n}{n!}$]

Proof of thm 2.3 Idea: To relate $\Lambda(s)$ and $\vartheta(t) := \theta(it)$ via an integral transform and use the transformation property of $\vartheta(t) = \frac{1}{\sqrt{t}}\,\vartheta\left(\frac{1}{t}\right)$

inside the integral
to analytically continue $\Lambda(s)$.

We start by collecting growth and decay properties of $\vartheta(t)$

$$\vartheta(t) \leq C t^{-1/2} \quad \text{as } t \to 0$$ (Follows from func. eqn)

and $$|\vartheta(t) - 1| \leq C e^{-\pi t}$$ for some $C>0$ and $\forall t \geq 1$.

Since $$2\sum_{n\geq 1} e^{-\pi n^2 t} \leq 2\sum_{n\geq 1} e^{-\pi n t} \leq C e^{-\pi t}$$
$t \geq 1$

The relation between Λ and ϑ is now

given by the fact that, for $\mathrm{Re}\, s > 1$

$$\boxed{\Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty (\vartheta(t) - 1)\, t^{\frac{s}{2}} \frac{dt}{t}}$$

This is based on the simple observations that

ⓐ $$\int_0^\infty e^{-\pi n^2 t}\, t^{\frac{s}{2}} \frac{dt}{t} = (\pi n^2)^{-s/2}\Gamma(s)$$
$$= \pi^{-s/2}\Gamma(s)\, n^{-s}$$

ⓑ $$\vartheta(t) - 1 = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

Using the estimates on $\vartheta(t)$ as $t \to 0$
as $t \to \infty$

one can justify the change of $\int$, Σ

to get $\frac{1}{2} \int_0^\infty (\vartheta(t) - 1)\, t^{s/2} \frac{dt}{t}$, $\mathrm{Re}\, s > 1$

$$= \sum_{n=1}^{\infty} \int_0^\infty e^{-\pi n^2 t}\, t^{s/2} \frac{dt}{t} = \pi^{-s/2} \Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s}$$
$$= \Lambda(s)$$

$$\boxed{\Lambda(s) = \frac{1}{2} \int_0^\infty (\vartheta(t) - 1)\, t^{s/2} \frac{dt}{t}} \quad , \mathrm{Re}\, s > 1$$

Now we'll see that we can make sense of the RHS for $s \in \mathbb{C}$.

Now
$$|\vartheta(t) - 1| < e^{-\pi t} \quad \text{for} \quad t \geq 1$$

$$\Rightarrow \quad \frac{1}{2} \int_1^\infty (\vartheta(t) - 1)\, t^{s/2} \frac{dt}{t} \quad \text{converges } \forall s.$$

Hence defines an analytic function $\forall s$.

On the other hand for

$$\frac{1}{2}\int_0^1 (\vartheta(t)-1)\, t^{s/2}\,\frac{dt}{t}$$ we use the

func'l eqn of $\vartheta(t) = t^{-1/2}\vartheta(1/t)$

This gives for $\vartheta(t)-1 = t^{-1/2}\vartheta(1/t)-1$

$$= t^{-1/2}(\vartheta(1/t)-1) + t^{-1/2} - 1$$

Hence $$\frac{1}{2}\int_0^1 (\vartheta(t)-1)\, t^{s/2}\,\frac{dt}{t}$$

$$= \frac{1}{2}\int_0^1 \left(t^{-1/2}\,(\vartheta(1/t)-1) + t^{-1/2} - 1\right) t^{s/2}\,\frac{dt}{t}$$

$$= \frac{1}{2}\int_0^1 (\vartheta(1/t)-1)\, t^{\frac{s-1}{2}}\,\frac{dt}{t} + \frac{1}{2}\int_0^1 t^{\frac{s-1}{2}}\,\frac{dt}{t} - \frac{1}{2}\int_0^1 t^{s/2}\,\frac{dt}{t}$$

$$= \frac{1}{2}\int_0^1 (\vartheta(1/t)-1)\, t^{\frac{s-1}{2}}\,\frac{dt}{t} + \frac{1}{2}\left.\frac{t^{\frac{s-1}{2}}}{(s-1)/2}\right|_0^1 - \frac{1}{2}\left.\frac{t^{s/2}}{s/2}\right|_0^1$$

$$= \frac{1}{2}\int_0^1 (\vartheta(1/t)-1)\, t^{\frac{s-1}{2}}\,\frac{dt}{t} + \frac{1}{s-1} - \frac{1}{s}$$

Now we make the change of variables
$u = 1/t \quad \frac{du}{u} = -dt/t$ we get

$$\frac{1}{2}\int_0^1 (\vartheta(1/t) - 1)\, t^{\frac{s-1}{2}}\, \frac{dt}{t} = \int_1^\infty (\vartheta(u) - 1)\, u^{\frac{1-s}{2}}\, \frac{du}{u}$$

Hence overall we have

$$\Lambda(s) = \frac{1}{2}\int_0^\infty (\vartheta(t) - 1)\, \frac{dt}{t} = \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (\vartheta(t) - 1)\left[t^{s/2} + t^{\frac{1-s}{2}}\right]\frac{dt}{t}\right]$$

Note the integral on the right defines an entire func. for all s due to the exponential decay of
$|\vartheta(t) - 1| < Ce^{-\pi t} \quad t \geq 1$

Hence $\Lambda(s)$ has a continuation to all s-plane which is holomorphic except for simple poles at $s = 1, 0$ w/ residues $1, -1$ resp.

Both $\frac{1}{s-1} - \frac{1}{s}$ and $\int_1^\infty (\vartheta(t) - 1)\left[t^{s/2} + t^{\frac{1-s}{2}}\right]\frac{dt}{t}$

are invariant under $s \to 1-s$ hence

$$\Lambda(s) = \Lambda(1-s)$$

$\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ has A.C. to all s except for simple poles at $s=0$, $s=1$.

Since $\Gamma(\frac{s}{2})$ has poles at $\frac{s}{2}=0,-1,-2\ldots$

$\zeta(s)$ does not have a pole at $s=0$

and must vanish at $s=-2,-4,-6,\ldots$

since $\Lambda(s)$ doesn't have poles at $s=-2,-4,\ldots$

These are called the trivial zeroes of $\zeta(s)$.

For $\mathrm{Re}\,s>1$,

$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ also has an infinite product expansion called Euler product

$\zeta(s) = \prod_{p \text{ primes}}(1-p^{-s})^{-1}$ which is an analytic statement

of fund. thm of arithmetic: Every positive integer is a unique product of prime powers.

Euler product $\Rightarrow$ $\zeta(s)\neq 0$ for $\mathrm{Re}\,s>1$

By func'l eqn $\zeta(s)\neq 0$ for $\mathrm{Re}\,s<0$

Riemann hypothesis: If $s \neq -2, -4, \ldots, s \in \mathbb{C}$ with $\zeta(s) = 0$ then $\operatorname{Re} s = \frac{1}{2}$

PNT Prime number theorem: let

$$\pi(x) := \#\{p \text{ prime} \mid p < x\}$$

Then

$$\lim_{x \to \infty} \frac{\pi(x)}{(x/\log x)} = 1$$

PNT is a consequence of

Thm If $\operatorname{Re} s = 1$ then $\zeta(s) \neq 0$