Homotopy, Homotopy Equivalence, Contractibility: Additional Exercises

1. (a) Show that any map $f\colon \mathbb{R}^n \to X$ is homotopic to the constant map.
   (b) Are any two maps $\mathbb{R}^n \to X$ homotopic?

2. Let $f, g\colon X \to \mathbb{C} \setminus \{0\}$ be such continuous maps that $|f(x) - g(x)| < |f(x)|$ for all $x \in X$. Show that f and g are homotopic.

3. Let X be any topological space, $x_0 \in S^n$ and $f\colon S^n \to X$ a continuous map. Then the following statements are equivalent:
   (a) i. f is homotopic to some constant map c.
       ii. There exists $F\colon B^{n+1} \to X$ such that $F|_{S^n} = f$.
       iii. $f \simeq c$ (rel x_0).
   (b) Maps $f, g\colon [-1,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$ are defined as follows:
   $$f(x) = (x, \sqrt{1-x^2}) \quad \text{and} \quad g(x) = (x, -\sqrt{1-x^2}).$$
   Show that $f \simeq g$, but $f \not\simeq g$ (rel $\{-1, 1\}$).

4. Which of the spaces $\mathbb{R}^2$, $\mathbb{R}^2 \setminus \{(0,0)\}$, $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$, $\mathbb{R} \times [0, \infty)$, $S^1 \times \mathbb{R}$ and Möbius band are homotopy equivalent?

5. Let $x_0 \in X$ be a strong deformation retract of X.
   (a) Show that for any neighborhood U of x_0 there exists a neighborhood $V \subset U$ of x_0 such that the inclusion $i\colon V \hookrightarrow U$ is homotopic to a constant.
   (b) Show that X is locally path connected in x_0.

6. Let $X = (([0,1] \times \{0\}) \cup (\{0\} \times [0,1])) \bigcup (\cup_{n=1}^{\infty}(\{\frac{1}{n}\} \times [0,1]))$.
   (a) Show that any point $x \in X$ is a deformation retract of X.
   (b) Find all points $x \in X$ that are strong deformation retracts of X.

7. Let
   $$T = ([0,1] \times \{0\}) \cup (\cup_{\mathbb{Q} \cap [0,1]}(\{q\} \times [0,q])) \subset \mathbb{R} \times \mathbb{R}$$
   and let $\hat{T}$ be the set obtained by reflecting T across the line $y = x$ and translating the result by $(0, -1)$. Let $X_0 = T \cup \hat{T}$ and for $n \in \mathbb{Z}$ let X_n be the set X_0 translated by (n, n). Let $X = \cup_{n \in \mathbb{Z}} X_n$.
   (a) Show that any point $x \in X$ is a deformation retract of X.
   (b) Show that no point $x \in X$ is a strong deformation retract of X.

8. (a) Let X be contractible. Show that any $f, g\colon X \to Y$ are homotopic for any path connected space Y.
   (b) Find a non-contractible space X for which any $f, g\colon X \to Y$ are homotopic for any path connected space Y.

9. If $f_0 \sim f_1\colon X \to Y$, then $M_{f_0} \sim M_{f_1}$ rel. $X \times \{0\}$.