Homological Algebra, Relative Homology, extra exercises

1. ('3 × 3 lemma.') Let

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

   be a commutative diagram of abelian groups. Assume that all three columns are exact. Prove that if the top two rows are exact, then so is the bottom.

2. ('Long exact homology sequence of a triple.') Let (C, B, A) be a 'triple,' so C is a space, B is a subspace of C, and A is a subspace of B. Show that there are maps $\partial\colon H_n(C, B) \to H_{n-1}(B, A)$ such that

$$\ldots \longrightarrow H_n(B, A) \xrightarrow{i_*} H_n(C, A) \xrightarrow{j_*} H_n(C, B) \longrightarrow H_{n-1}(B, A) \xrightarrow{i_*} \ldots$$

   is exact, where $i\colon (B, A) \to (C, A)$ and $j\colon (C, A) \to (C, B)$ are the inclusions of pairs.

3. Suppose that

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \cong\downarrow & & \downarrow & & \\ \ldots & \longrightarrow & A'_n & \longrightarrow & B'_n & \longrightarrow & C'_n & \longrightarrow & A'_{n-1} & \longrightarrow & \ldots \end{array}$$

   is a "ladder": a map of long exact sequences. So both rows are exact and each square commutes. Suppose also that every third vertical map is an isomorphism, as indicated. Prove that these data naturally determine a long exact sequence

$$\ldots \longrightarrow A_n \longrightarrow A'_n \oplus B_n \longrightarrow B'_n \longrightarrow A_{n-1} \longrightarrow \ldots$$

4. Let $A \subset X$ be a retract of X. Prove that $\mathrm{H}_n(X) \cong \mathrm{H}_n(A) \oplus \mathrm{H}_n(X, A)$ for every $n \in \mathbb{N}_0$.

5. Let D be a 2-disc with k open discs removed. Compute the homology of the pair $(D, \partial D)$.