# Additional sheet 4

**Problem 1:** Let $A$ be a finite set of points in the two-dimensional torus $T$. Compute the homology groups of $T \setminus A$ and $T/A$.

**Problem 2:** Let $X_1, \dots, X_k$ be an open covering of $X$ and $Y_1, \dots Y_k$ and open covering of $Y$. If $f : X \to Y$ is a continuous map such that $f(X_i) \subset Y_i$ for all $i$, and the restrictions

$$\bigcap_{i \in A} X_i \longrightarrow \bigcap_{i \in A} Y_i$$

induce isomorphisms in homology, for all finite sets $A \subset \{1, \dots, k\}$. Show that $f$ induces an isomorphism in homology.

**Problem 3:** If $f : A \to B$ and $g : B \to C$ are homomorphisms of abelian groups, show that there is an exact sequence

$$0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(gf) \to \operatorname{coker}(g) \to 0$$

**Problem 4:** Let $f : X \to Y$ be a continuous map of topological spaces. Remember that we defined the *mapping cone* of $f$ to be

$$C_f = \frac{X \times [0,1] \sqcup Y}{(x_1, 0) \sim (x_2, 0) \text{ and } (x, 1) \sim f(x)}.$$

Show that there is a long exact sequence

$$\dots \to \tilde{H}_i(X) \xrightarrow{f_*} \tilde{H}_i(Y) \to \tilde{H}_i(C_f) \to \tilde{H}_{i-1}(X) \to \dots$$

**Problem 5:** Let $X$ be the interval $[0,1]$ and $A = \{\frac{1}{n} \mid n \in \mathbb{N}_{>0}\} \cup \{0\}$.

a) Show that the group $H_1(X/A)$ is uncountable.
(Hint: use that $X/A$ is isomorphic to the subset

$$\bigcup_{n>0} \left\{ z \in \mathbb{C} : \left| z - \frac{1}{n} \right| = \frac{1}{n} \right\}$$

of the plane, to construct a surjective homomorphism $\pi_1(X/A) \to \prod_{n>0} \mathbb{Z}$, and use the Hurewicz theorem).

b) Show that $\tilde{H}_1(X/A)$ does not fit into an exact sequence

$$\tilde{H}_1(A) \to \tilde{H}_1(X) \to \tilde{H}_1(X/A) \to \tilde{H}_0(A) \to \tilde{H}_0(X)$$

and therefore is not isomorphic to $H_1(X, A)$.