# Additional sheet 5

**Problem 1:** Let $M$ be the Möbius band, and let $f : \partial M = \mathbb{S}^1 \to X$ be a continuous map. Compute the homology groups of $X \cup_f M$ in the following cases:

*a)* $f : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ is the map $z \mapsto (z^n, z^m)$, where $n, m \in \mathbb{Z}$.

*b)* $f : \mathbb{S}^1 \to \mathbb{RP}^2$ is the canonical embedding of $\mathbb{RP}^1$ in $\mathbb{RP}^2$ as the line at infinity.

*c)* $X = \Sigma_g$, where $D \subset \Sigma_g$ is a disk and $f : \mathbb{S}^1 \to \partial D$.

**Problem 2:** Let $A$ be a $2 \times 2$ matrix with integral coefficients. Considering the 2-dimensional torus $T$ as $\mathbb{R}^2/\mathbb{Z}^2$, $A$ induces a map $f_A : T \to T$. Show that

*a)* $f_{A,*} : H_1(T) \to H_1(T)$ is given by multiplication by $A$.

*b)* $f_{A,*} : H_2(T) \to H_2(T)$ is given by multiplication by $\det(A)$.

*c)* (Hard, but $n = 3$ is still reasonable. Don't get frustrated if you cannot get this one.) generalise this to maps from the $n$-dimensional torus to itself given by integral $n \times n$ matrices.

**Problem 3:** Consider $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ as the equator. Let $\sim$ be the equivalence relation in $\mathbb{S}^{n-1}$ given by $x \sim -x$, and let $X = \mathbb{S}^n / \sim$ be the quotient space. Compute the homology groups of $X$.

**Problem 4:** Let $X$ be any topological space. Show that

$$H_i(X \times \mathbb{S}^1) = \begin{cases} H_i(X) \times H_{i-1}(X) & \text{if } i \geq 1 \\ H_0(X) & \text{if } i = 0 \end{cases}$$

(Harder) Generalise this to $X \times \mathbb{S}^n$

**Problem 5:**

*a)* Use problem 5 to show that the spaces $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^3$ have the same homology groups

*b)* Show that the universal covering space of $\mathbb{S}^1 \times \mathbb{S}^2$ is $\mathbb{R} \times \mathbb{S}^2$, and the universal covering space of $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^3$ is the space obtained by attaching a copy of $\mathbb{S}^2 \vee \mathbb{S}^3$ at each integer.

*c)* Use *b)* to show that $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^3$ are not homeomorphic.

**Note**: I will not have time to do every problem in this sheet, but this way you also have something to practice with during the holidays. My plan is to do problems 1b), 2, 3, 4, but send an email to aitor.iribarlopez@math.ethz.ch if you want me to solve some other problem instead, specially if you have tried to solve it and got stuck at something.

**Hints**: (I will update this after Tuesday)

P2: For $a$), use the Hurewicz theorem and the universal cover of $T$, For $b$), use a Delta structure, or a CW structure on the torus and compute directly the image of the generator of $H_2(T)$.

P4: Start with a Mayer-Vietoris sequence to get short exact sequences of the form

$$0 \to H_i(X) \to H_i(X \times \mathbb{S}^1) \to H_{i-1}(X) \to 0$$

And one has to find a splitting $H_i(X \times \mathbb{S}^1) \to H_i(X)$.
For the generalisation, maybe you can have a look at the Kunneth Theorem for inspiration, but the proof as in the case $n = 1$ still works. The only thing you need to know about the symbol $\otimes$ is that $A \otimes \mathbb{Z} = A$ and $A \otimes 0 = 0$ for any abelian group.