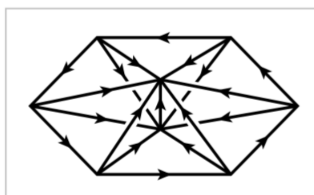
Homework 2

1. (a) Construct a Δ-complex structure on the torus T, and use it to compute the simplicial homology groups of T.
   (b) Construct a Δ-complex structure on the Klein bottle K, and use it to compute the simplicial homology groups of K.
   (c) Construct a Δ-complex structure on the projective plane $\mathbb{RP}^2$, and use it to compute the simplicial homology groups of $\mathbb{RP}^2$.

   Though we have not proved that the homology groups are independent of the choice of Δ-complex structure we choose, you may use this fact.

2. Compute the simplicial homology groups of the Δ-complex obtained from $n+1$ 2-simplices $\Delta_0^2, \ldots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.

3. Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \ldots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1} , subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions $0, 1, 2, 3$ are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$, respectively.



4. Suppose that X is a path-connected space and let $f\colon X \to X$ be a map. Prove that the induced map $f_*\colon H_0(X) \to H_0(X)$ is the identity. What happens if X is not path-connected?

5. (optional) We mentioned in class that homology has the added benefit of being easy to compute. Software exists to compute it for a special type of Δ-complexes called simplicial complexes. Formally, a **geometric simplicial complexes** $\mathcal{K}$ is a set of simplices (in Euclidean space) that satisfies the following conditions:

   (a) Every face of a simplex from $\mathcal{K}$ is also in $\mathcal{K}$.
   (b) The non-empty intersection of any two simplices σ_1 and σ_2 in $\mathcal{K}$ is a face of both σ_1 and σ_2.

   Geometric simplicial complexes can be represented by **abstract simplicial complexes** that only retain the information about the connections (edges, triangles, etc) between the vertices, but not the coordinates:

   Definition An **abstract simplicial complex** is given by the following data.

   - A set V of vertices or 0-simplices.
   - For each $k \geq 1$, a set of k-simplices $\sigma = [v_0, v_1, \ldots, v_k]$, where $v_i \in V$.
   - Each k-simplex has $k+1$ faces obtained by deleting one of the vertices. The following membership property must be satisfied: if σ is in the simplicial complex, then all faces of σ must be in the simplicial complex.

One must also make adjustments in the construction of the chain complex – instead of forming the chain groups with coefficients from $\mathbb{Z}$, we take them from some finite field $\mathbb{Z}_p$ (very often $p = 2$). With this the chain groups become vector spaces and boundary maps linear maps. All the computations can be carried out using linear algebra.

A good software to start with is Javaplex, available here. To use it you will first need to download Matlab. Use chapter 1 of the accompanying Javaplex tutorial (available here) to install Javaplex on your computer.

(a) Read the first 6 pages of the Javaplex tutorial (up to section 3.2).

(b) Compute the homology groups of the house example from class over $\mathbb{Z}/2\mathbb{Z}$. Compare the results to ours from class.

(c) Find a simplicial complex structure on the torus and determine its homology groups over $\mathbb{Z}/2\mathbb{Z}$.

(d) Find a simplicial complex structure on Klein bottle and determine its homology groups over $\mathbb{Z}/2\mathbb{Z}$.

(e) Compare the homology groups of the Klein bottle and the torus. What do you notice? Compute homology groups of both over $\mathbb{Z}/3\mathbb{Z}$ using Javaplex. Are you able to distinguish between them?