# Homework 3

1. (a) Compute the fundamental group of $T \setminus \{p\}$, where $T$ is the torus and $p$ is any point in $T$.
   (b) Compute the fundamental group of $K \setminus \{p\}$, where $K$ is the Klein bottle and $p$ is any point in $K$.
   (c) Use Van Kampen's Theorem to compute $\pi_1(T)$ and $\pi_1(K)$ (Hint: Use 1 & 2).
   (d) Use the Hurewicz Theorem to compute $\mathrm{H}_1(T)$ and $\mathrm{H}_1(K)$.

2. Let $f\colon (X, x_0) \to (Y, y_0)$ be a map of pointed spaces and consider the induced maps $f_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $f_*\colon \mathrm{H}_1(X) \to \mathrm{H}_1(Y)$. Prove commutativity of the diagram

$$\begin{array}{ccc}
\pi_1(X, x_0) & \xrightarrow{f_\#} & \pi_1(Y, y_0) \\
\downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_Y} \\
\mathrm{H}_1(X) & \xrightarrow{f_*} & \mathrm{H}_1(Y)
\end{array}$$

   where $\phi_X$ and $\phi_Y$ are the Hurewicz homomorphisms.

3. Verify that $f \sim g$ implies $f_* = g_*$ for induced homomorphisms of reduced homology groups.

4. Prove the snake lemma.

   Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & A' & \longrightarrow & B' & \xrightarrow{p} & C' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A & \xrightarrow{i} & B & \longrightarrow & C & &
\end{array}$$

   where the rows are exact. Then there is an exact sequence relating the kernels and cokernels of $f, g$, and $h$, where $\partial$ is a homomorphism $\partial\colon \ker h \to \operatorname{coker} f$, known as the connecting homomorphism.

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\partial} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h.$$

5. For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces $(X, A)$, the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $\mathrm{H}_n(X, A) = 0$ for all $n$.