Homework 4

1. Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about the case of short exact sequences $0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$?

2. Let $A \subset X$ be a non-empty subset and assume that $\tilde{\mathrm{H}}_n(A) = 0$ for all n (that is, A is acyclic). Prove that $\mathrm{H}_n(X, A) \cong \tilde{\mathrm{H}}_n(X)$ for all n.

3. (a) Show that $\mathrm{H}_0(X, A) = 0$ iff A meets each path-component of X .
   (b) Show that $\mathrm{H}_1(X, A) = 0$ iff $\mathrm{H}_1(A) \to \mathrm{H}_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

4. Show that for the subspace $\mathbb{Q} \subset \mathbb{R}$, the relative homology group $\mathrm{H}_1(\mathbb{R}, \mathbb{Q})$ is free abelian and find a basis.

5. A pair of topological spaces (X, A) is a good pair if A is a nonempty closed subspace and there exists an open neighborhood V of A in X that strongly deformation retracts onto A.
   (a) Let $q\colon X \to X/A$ be the quotient map. Show that the diagram

$$\begin{array}{ccccc}
\tilde{\mathrm{H}}_p(X, A) & \longrightarrow & \tilde{\mathrm{H}}_p(X, V) & \longleftarrow & \tilde{\mathrm{H}}_p(X - A, V - A) \\
\downarrow q_* & & \downarrow q_* & & \downarrow q_* \\
\tilde{\mathrm{H}}_p(X/A, A/A) & \longrightarrow & \tilde{\mathrm{H}}_p(X/A, V/A) & \leftarrow & \tilde{\mathrm{H}}_p(X/A - A/A, V/A - A/A)
\end{array}$$

   is commutative.
   (b) Show that the horizontal arrows are isomorphisms.
   (c) Show that the vertical arrow on the right is an isomorphism.
   (d) Prove from (a), (b), (c) that, if (X, A) is a good pair, it holds $\mathrm{H}_p(X, A) = \tilde{\mathrm{H}}_p(X/A)$ for each $p > 0$.
   (e) Conclude the following theorem.
   Theorem: Let (X, A) be a good pair. Then there is a long exact sequence

$$\ldots \longrightarrow \tilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \tilde{\mathrm{H}}_n(X) \xrightarrow{j_*} \tilde{\mathrm{H}}_n(X/A) \to \tilde{\mathrm{H}}_{n-1}(A) \xrightarrow{i_*} \ldots \longrightarrow \tilde{\mathrm{H}}_0(X/A) \longrightarrow 0$$

   where i is the inclusion A and j is the quotient map $X \to X/A$.

6. Define the unreduced suspension ΣX of a space X to be the quotient space of $[0, 1] \times X$ obtained by identifying $\{0\} \times X$ and $\{1\} \times X$ to points. Show that there is a natural isomorphism $\tilde{\mathrm{H}}_n(X) \to \tilde{\mathrm{H}}_{n+1}(\Sigma X)$. Here natural means that for a map $f\colon X \to Y$, and its suspension $\Sigma f\colon \Sigma X \to \Sigma Y$ the following diagram commutes:

$$\begin{array}{ccc}
\tilde{\mathrm{H}}_n(X) & \xrightarrow{\cong} & \tilde{\mathrm{H}}_{n+1}(\Sigma X) \\
f_* \downarrow & & \downarrow (\Sigma f)_* \\
\tilde{\mathrm{H}}_n(Y) & \xrightarrow{\cong} & \tilde{\mathrm{H}}_{n+1}(\Sigma Y)
\end{array}$$

   Hint: Consider the two cones $C_+X := \{[t, x] \in \Sigma X \mid t \geq \frac{1}{2}\}$ and $C_-X := \{[t, x] \in \Sigma X \mid t \leq \frac{1}{2}\}$.