Homework 7

1. Recall that $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/_\sim$, where $\underline{x} \sim \lambda\underline{x}$ for all $0 \neq \lambda \in \mathbb{R}$.

   (a) Find explicit homeomorphisms between $\mathbb{R}P^n$ and the following two spaces: $S^n/_\sim$, where $x \sim -x$ for all $x \in S^n, B^n/_\sim$, where $x \sim -x$ for all $x \in \partial B^n$.

   (b) Endow $\mathbb{R}P^n$ with the structure of a CW-complex with precisely one k-cell in each dimension $0 \leq k \leq n$ and no cells in dimension higher than n.

   (c) Calculate the cellular homology of $\mathbb{R}P^n$.

2. Let $G \subset \mathbb{R}^2$ be a finite connected planar graph with v vertices, e edges and f faces. (A face is a region in $\mathbb{R}^2$ that is bounded by edges. The infinitely large region outside of the graph is also a face, called the outer face.) Prove the Euler formula:

$$v - e + f = 2.$$

   Hint: Check out the definition of the Euler Characteristic in Hatcher's AT, page 146.

3. The 3-torus is the quotient space $T^3 = \mathbb{R}^3/\mathbb{Z}^3 \approx S^1 \times S^1 \times S^1$. Find a CW-structure on T^3 and use it to compute homology groups $H_p(T^3)$ for all p.

4. Consider the space X which is the union of the unit sphere $S^2 \subset \mathbb{R}^3$ and the line segment between the north and south poles.

   (a) Give X a CW-structure and use it to compute $H_p(X)$ for all p.

   (b) Use that X is homotopy equivalent to $S^2 \vee S^1$ to give an easier computation of $H_p(X)$ for all p.

5. Let C be the circle on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ which is the image, under the covering map $\mathbb{R}^2 \to T^2$, of the line $px = qy$. Define $X = T^2/C$, the quotient space obtained by identifying C to a point. Compute $H_p(X)$ for all p.

6. Compute $H_p(\mathbb{R}P^n/\mathbb{R}P^m)$ for $m < n$, using cellular homology and equipping $\mathbb{R}P^n$ with the standard CW-structure with $\mathbb{R}P^m$ as its m-skeleton.