Solutions to problem set 1

Notation. $I := [0,1]$; we omit $\circ$ in compositions: $fg := f \circ g$.

1. By [Bredon, Prop. I.14.5], contractibility of X means that id_X is homotopic to a map whose image is a singleton $\{x_0\} \in X$; that is, there exists a map $h : X \times I \to X$ such that $h(x,0) = x$ and $h(x,1) = x_0$ for all $x \in X$. Consider now the map $h' = r \circ h|_{A \times I} : A \times I \to A$. For $a \in A$, it satisfies $h'(a,0) = r(h(a,0)) = r(a) = a$ by the defining property of retraction, and $h'(a,1) = r(x_0)$; in other words, id_A is homotopic to a map with image $\{r(x_0)\} \subset A$, and hence A is contractible (again by [Bredon, Prop. I.14.5]).

2. Recall that $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. We claim that the expression
$$(x,t) \mapsto \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|},$$
yields a well-defined map $\phi : X \times I \to S^n$. Pretending that this is true, note that $\phi(x,0) = f(x)$ and $\phi(x,1) = g(x)$ for all $x \in X$, and hence f and g are homotopic.

   To prove well-definedness, we must show that the denominator never vanishes. To do so, we assume the contrary, i.e., that there exists some $(x,t) \in X \times I$ such that $(1-t)f(x) + tg(x) = 0$. This is equivalent to $(1-t)f(x) = -tg(x)$, from which we obtain $(1-t) \cdot \|f(x)\| = t \cdot \|g(x)\|$; since $\|f(x)\| = \|g(x)\| = 1$, it follows that $1 - t = t$, and hence $t = \frac{1}{2}$. Inserting this into the first equation, we obtain $f(x) = -g(x)$, which contradicts our assumption.

3. Using the assumptions $fg \simeq \mathrm{id}_Y$ and $hf \simeq \mathrm{id}_X$, we obtain
$$fh \;=\; fh\,\mathrm{id}_Y \;\simeq\; fhfg \;\simeq\; f\,\mathrm{id}_X\,g \;=\; fg \;\simeq\; \mathrm{id}_Y.$$
Hence h is in fact a homotopy inverse of f, and thus f is a homotopy equivalence. (Similarly, one can show that g is a homotopy inverse of f.)

4. (a) Let $q : Y \sqcup (X \times I) \to M_f$ be the quotient map. By definition of the quotient topology, a subset $V \subset M_f$ is open if and only if $q^{-1}(V) \subset Y \sqcup (X \times I)$ is open. Hence we get the following:
$$\begin{aligned}
&\phi \text{ is continuous} \\
&\iff \forall U \subset Z \text{ open}: \ \phi^{-1}(U) \subset M_f \text{ is open} \\
&\iff \forall U \subset Z \text{ open}: \ q^{-1} \circ \phi^{-1}(U) \subset Y \sqcup (X \times I) \text{ is open} \\
&\iff \forall U \subset Z \text{ open}: \ \begin{cases} \phi_{X \times I}^{-1}(U) = (\phi \circ q \circ i_{X \times I})^{-1}(U) \subset X \times I \\ \phi_Y^{-1}(U) = (\phi \circ q \circ i_Y)^{-1}(U) \subset Y \end{cases} \quad \text{are open} \\
&\iff \phi_{X \times I} \text{ and } \phi_Y \text{ are continuous.}
\end{aligned}$$

   (b) By definition of the maps that are involved we have $ri_X = f$ and hence the diagram is commutative.

   Let $F : M_f \times I \to M_f$ be defined by:
$$\begin{cases} F([x,t],t') = [x, t(1-t')], \text{ for } x \in X, t, t' \in I \\ F([y],t') = [y], \text{ for } y \in Y, t' \in I. \end{cases}$$
Then F is a homotopy rel Y from id_{M_f} to $i_Y \circ r$. (The continuity of F follows from the fact that $M_f \times I \approx M_{f \times id_I}$ combined with the result of exercise 4(a).)

(c) For a very nice solution see the proof of Proposition 7.46 (p. 206) in John Lee's book *Introduction to Topological Manifolds.*

By definition a subspace $A \subset B$ is a deformation retract if there is a retraction $r\colon B \to A$ which is a right homotopy inverse of the inclusion map $i : A \hookrightarrow B$. Explicitely, this means that $r \circ i = id_A$ and $i \circ r = \simeq id_B$. In particular, if $A \subset B$ is a deformation retract, then A and B have the same homotopy type. So if X and Y can be embedded as weak deformation retracts of the same space Z, then X and Y both have the same homotopy type as Z and hence they are homotopy equivalent.

Conversely, suppose that $f : X \to Y$ is a homotopy equivalence. We will show that both X and Y are deformation retracts of M_f.

The retraction $r\colon M \to Y$ from 4(b) satisfies $r \circ i_y$ and $i_y \circ r \simeq id_{M_f}$. This shows that Y is a (strong) deformation retract of M_f.

Again by 4(b) we have $f = r \circ i_X$ and thus $X \xrightarrow{i_X} M_f \xrightarrow{r} Y$ is a homotopy equivalence. Since r is also a homotopy equivalence, it follows that i_X is a homotopy equivalence as well. Now let $g : M_f \to X$ be a homotopy inverse of i_X, i.e. $g \circ i_X \simeq id_X$ and $i_X \circ g \simeq id_{M_f}$. The idea is to modify g to construct a *retraction* $q\colon M_f \to X$ with $i_X \circ q \simeq id_{M_f}$. Denote by $G : X \times I \to X$ a homotopy $G : g \circ i_X \simeq id_X$. Define the homotopy $H : M_f \times I \to X$ by $H([y], t') = g([y])$ for $y \in Y$ and $t' \in I$ and

$$H([x,t],t') = \begin{cases} g\left(\left[x, \frac{2t}{2-t'}\right]\right), & 0 \le t' \le 2(1-t) \le 2,\ x \in X,\ t, t' \in I \\ G\left(x, \frac{2t-(2-t')}{t}\right), & 0 \le 2(1-t) \le t' \le 1,\ x \in X,\ t, t' \in I. \end{cases}$$

First note that H is well-defined as a map on $(Y \sqcup (X \times I)) \times I$ because for $t' = 2(1-t)$ one has

$$g\left(\left[x, \frac{2t}{2-t'}\right]\right) = g([x,1]) = G(x,0) = G\left(x, \frac{2t-(2-t')}{t}\right).$$

Moreover, H descends to a well-defined map on $M_f \times I$ because

$$H([x,0],t') = g([x,0]) = g([f(x)]) = H([f(x)],t').$$

Consider $q\colon M_f \to X$, $z \mapsto H(z,1)$. q is a retraction: $q \circ i_X(x) = H([x,1],1) = G(x,1) = x$, so $q \circ i_X = id_X$. Moreover, $i_X \circ H$ is a homotopy from $i_X \circ g$ to $i_X \circ q$. Thus $id_{M_f} \simeq i_X \circ g \simeq i_X \circ q$. We conclude that X is also embedded as a deformation retract of M_f.

5. We view $\mathbb{R}P^2$ as $D^2/\sim$, where $\sim$ is the equivalence relation that identifies antipodal points on the boundary. That is, $x, y \in D^2$ satisfy $x \sim y$ if and only if $x = y$, or x and y both lie on $S^1 \subset D^2$ and satisfy $x = -y$.

   Consider the map $F : S^1 \sqcup (S^1 \times I) \to \mathbb{R}P^2$ defined on S^1 by $e^{2\pi it} \mapsto [e^{\pi it}]$ for $t \in [0,1]$, and on $S^1 \times I$ by $(e^{2\pi it}, s) \mapsto [(1-s)e^{2\pi it}]$. (Note that the first part is well-defined and continuous, because $e^{\pi i0} = 1 \sim -1 = e^{\pi i1}$ and hence $[e^{\pi i0}] = [e^{\pi i1}] \in \mathbb{R}P^2$.)

   Observe that F descends to a well-defined map $F' : M_f \to \mathbb{R}P^2$, because $F(f(e^{2\pi it})) = F(e^{4\pi it}) = [e^{2\pi it}] = F(e^{2\pi it}, 0)$, and moreover F' is clearly surjective. Since F' maps all of $S^1 \times \{1\}$ to a single point, namely $[0] \in \mathbb{R}P^2$, it descends further to a map $F'' : C_f \to \mathbb{R}P^2$. F'' is still surjective. Note that F'' is also injective, since F' is injective on $M_f \setminus (S^1 \times \{1\})$ with image in $\mathbb{R}P^2 \backslash \{0\}$.

   Since C_f is compact and $\mathbb{R}P^2$ is Hausdorff, we conclude using [Bredon, Theorem I.7.8] that F'' is a homeomorphism.