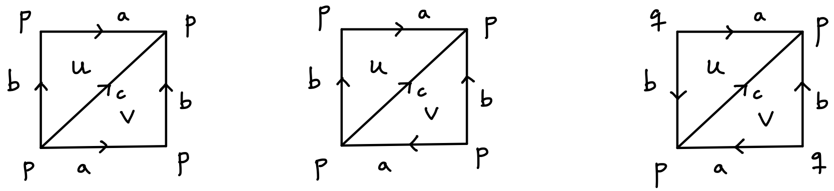
Problem 1:



torus Klein bottle $\mathbb{RP}^2$

torus: $\mathbb{Z}u \oplus \mathbb{Z}v \xrightarrow{\partial_2} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{\partial_1} \mathbb{Z}p$

$u \longmapsto a+b-c \quad | \quad a,b,c \longrightarrow 0$

$v \longmapsto a+b-c \quad |$

$$\Rightarrow H_k^{\Delta}(T) = \begin{cases} \mathbb{Z} & \text{for } k=0,2 \\ \mathbb{Z}^2 & \text{for } k=1 \\ 0 & \text{otherwise} \end{cases}$$

Klein bottle: $\mathbb{Z}u \oplus \mathbb{Z}v \xrightarrow{\partial_2} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{\partial_1} \mathbb{Z}p$

$u \longrightarrow a+b-c \quad | \quad a,b,c \longrightarrow 0$

$v \longrightarrow a+c-b \quad |$

$$\Rightarrow H_k^{\Delta}(K) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & k=1 \\ \mathbb{Z} & \text{for } k=0 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{RP}^2$: $\mathbb{Z}u \oplus \mathbb{Z}v \xrightarrow{\partial_2} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{\partial_1} \mathbb{Z}p \oplus \mathbb{Z}q$

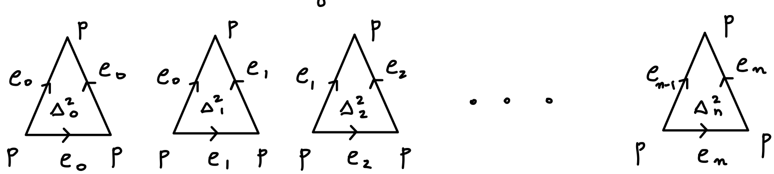
$u \longrightarrow b+c-a \quad | \quad a,b \longrightarrow p-q$

$v \longrightarrow a+c-b \quad | \quad c \longrightarrow 0$

$$\Rightarrow H_k^{\Delta}(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}/2 & \text{for } k=1 \\ \mathbb{Z} & \text{for } k=0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 2:

Give names to the identifications:



$\mathbb{Z}\langle \Delta_0^2, \dots, \Delta_n^2 \rangle \xrightarrow{\partial_2} \mathbb{Z}\langle e_0, \dots, e_n \rangle \xrightarrow{\partial_1} \mathbb{Z}p$

$\Delta_0^2 \longrightarrow e_0 \quad | \quad e_i \longmapsto 0$

$(i>0) \quad \Delta_i^2 \longrightarrow 2e_i - e_{i-1} \quad |$

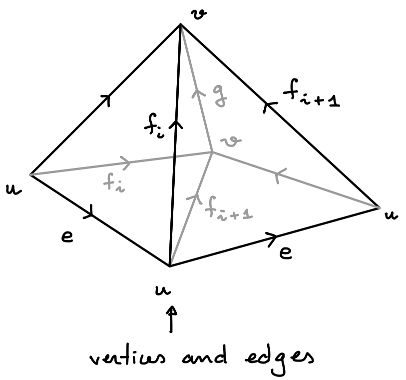
so ∂_2 is given by the square matrix $\begin{pmatrix} 1 & -1 & & 0 \\ 0 & 2 & \ddots & \\ & & \ddots & -1 \\ 0 & & & 2 \end{pmatrix}$, whose determinant is non-zero, so ∂_2 is injective and $H_2^\Delta(X) = 0$. Since $\partial_1 = 0$,

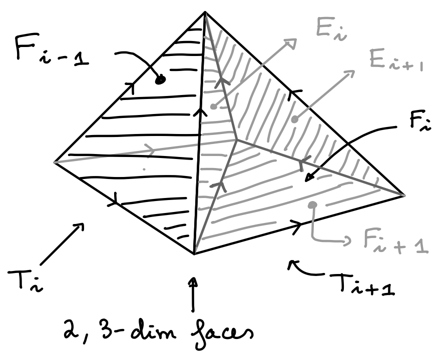
$$H_1^\Delta(X) \cong \frac{\mathbb{Z}\langle e_0, \dots, e_n\rangle}{\langle e_0, 2e_0 - e_1, \dots, 2e_{n-1} - e_n\rangle} \cong \frac{\mathbb{Z}\langle e_n\rangle}{2^n e_n} \cong \mathbb{Z}/2^n$$

and finally, $H_0^\Delta = \frac{\mathbb{Z}}{\text{im } \partial_1} = \mathbb{Z}$. For $k > 2$, $H_k^\Delta(X) = 0$.

Problem 3:

This is how the identification looks like:





$$\mathbb{Z}\langle T_1, \dots, T_n\rangle \xrightarrow{\partial_3} \mathbb{Z}\langle F_1, E_1, \dots, F_n, E_n\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle e, g, f_1, \dots, f_n\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle u, v\rangle$$

$T_i \longmapsto E_i - E_{i-1} + F_{i-1} - F_i$ | $F_i \longrightarrow e + f_{i+1} - f_i$ | $e \longrightarrow 0$

$E_i \longrightarrow f_{i+1} + g - f_i$ | $g \longrightarrow 0$

$f_i \longrightarrow v - u$

all indexes are mod n

$\longrightarrow \ker(\partial_3) = \langle T_1 + \dots + T_n\rangle$. Proof: clearly $T_1 + \dots + T_n$ is in the kernel, consider the composition $\pi \circ \partial_3 : \bigoplus_{i=1}^n \mathbb{Z}T_i \longrightarrow \bigoplus_{i=1}^n \mathbb{Z}E_i \oplus \mathbb{Z}F_i \longrightarrow \bigoplus_{i=1}^n \mathbb{Z}E_i$ whose matrix is $\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & & & & -1 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}$, and its kernel is

$$\{(x_1, \dots, x_n) \mid x_1 - x_2 = \dots = x_n - x_1 = 0\} = \langle (1, \dots, 1)\rangle$$ $\boxtimes$ Therefore,

$H_3^\Delta(X) = \ker(\partial_3) \cong \mathbb{Z}$

$\longrightarrow$ If $c = \sum_{i=1}^{n} x_i F_i + y_i E_i \in \ker(\partial_2)$,

$$0 = \partial_1(c) = \left(\sum_{i=1}^{n} x_i\right) e + \left(\sum_{i=1}^{n} y_i\right) g + \sum_{i=1}^{n} (x_i - x_{i+1} + y_i - y_{i+1}) f_i$$

In particular, $x_i + y_i = x_{i-1} + y_{i-1} = \dots =: \alpha$, but since also

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0,$$

$n\alpha = 0 \Rightarrow \alpha = 0$, and so $c = \sum_{i=1}^{n} x_i (F_i - E_i)$, with $\sum x_i = 0$. Recall that

$\bigstar$ [The kernel of the map $m : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ given by $m(z_1, \dots, z_n) = z_1 + \dots + z_n$ is freely generated by $(1,0,\dots,0,-1), (-1,1,0,\dots,0), \dots, (0,\dots,0,1,-1,0)$

This means that we can write c as a linear combination of

$F_1 - E_1 - (F_n - E_n)$, $F_2 - E_2 - (F_1 - E_1)$, $\dots$, and so $c \in \operatorname{im}(\partial_3)$. Therefore $H_2^{\Delta}(X) = 0$.

$\longrightarrow$ $\ker(\partial_1) \overset{\text{freely generated}}{=} \langle g, e, f_1 - f_2, \dots, f_{n-1} - f_n \rangle$. Proof: If $w = v - u$, we can regard $\partial_1 : \mathbb{Z}g \oplus \mathbb{Z}e \oplus \mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_n \longrightarrow \mathbb{Z}w$, given by the matrix $(0,0,1,\dots,1)$, its kernel is $\mathbb{Z}g \oplus \mathbb{Z}e \oplus \{z_1 f_1 + \dots + z_n f_n \mid \sum_{i=1}^{n} z_i = 0\}$, and we use $\bigstar$. $\boxtimes$

Let $d_i = f_i - f_{i+1}$ for $i = 1, \dots, n-1$. Then $f_n - f_1 = -\sum_{i=1}^{n-1} d_i$, and so

$$H_1^{\Delta}(X) \cong \frac{\mathbb{Z}\langle e, g, d_1, \dots, d_{n-1} \rangle}{\left\langle \begin{matrix} e + d_1, g + d_1, \dots, e + d_{n-1}, \\ g + d_{n-1}, e - \sum d_i, \\ g - \sum d_i \end{matrix} \right\rangle} \overset{e = g = \sum d_i}{\cong} \frac{\mathbb{Z}\langle d_1, \dots, d_{n-1} \rangle}{\left\langle \begin{matrix} 2d_1 + d_2 + \dots \\ d_1 + 2d_2 + \dots \\ \vdots \\ d_1 + \dots + 2d_n \end{matrix} \right\rangle} \overset{d := \sum d_i}{\cong} \frac{\mathbb{Z}\langle d_1, \dots, d_{n-1}, d \rangle}{\left\langle \begin{matrix} d_i + d \quad (i = 1, \dots, n-1) \\ d - d_1 - \dots - d_{n-1} \end{matrix} \right\rangle} \overset{d_i = -d}{\cong}$$

$$\cong \frac{\mathbb{Z}\langle d \rangle}{\langle \underbrace{d + \dots + d}_{n \text{ times}} \rangle} \cong \mathbb{Z}/n$$

$\longrightarrow$ $H_0^{\Delta}(X) = \dfrac{\mathbb{Z}\langle u, v \rangle}{\langle v - u \rangle} \cong \mathbb{Z}$. $\qquad$ $\longrightarrow$ If $k \geq 3$, $H_k^{\Delta}(X) = 0$

Problem 4:

Let $\mathrm{Path}(X) = \left\{ \begin{array}{l} \text{path-connected} \\ \text{components of } X \end{array} \right\}$, and pick $x_P \in P$ for all $P \in \mathrm{Path}(X)$.

Then the natural map $\bigoplus_{\mathrm{Path}(X)} \mathbb{Z}[x_P] \xrightarrow{\phi} H_0(X)$ is an isomorphism.

Any map $f : X \longrightarrow X$ gives a map $\tilde{f} : \mathrm{Path}(X) \longrightarrow \mathrm{Path}(X)$:

$$\tilde{f}(P) = \text{the unique component containing } f(P)$$

This gives a map $\bigoplus_{\mathrm{Path}(X)} \mathbb{Z}[x_P] \longrightarrow \bigoplus_{\mathrm{Path}(X)} \mathbb{Z}[x_P]$ by permuting the factors,

and this is f_* under the isomorphism ϕ. If $|\mathrm{Path}(X)| = 1$, $\tilde{f} = id \Rightarrow f_* = id$.