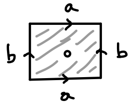
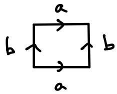
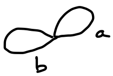
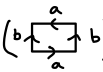
Problem 1

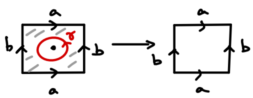
a) $T\setminus\{p\}$ can be written as , which is homotopy equivalent to  $\cong$ , so $\pi_1(T\setminus\{p\}) \cong \pi_1(S^1 \vee S^1) = \mathbb{Z}a * \mathbb{Z}b$

b) By the same argument, $\pi_1(K\setminus\{p\}) = \pi_1\big($  $\big) = \mathbb{Z}a * \mathbb{Z}b$

c) Let $U = T\setminus\{p\}$, $V = B(p,\varepsilon)$, a small disk around p, then $U, V, U\cap V$ are path connected, so we can apply the Seifert-van Kampen theorem. Since V is contractible and $U\cap V$ is homotopic to S^1, the theorem says that

$$\pi_1(T) \cong \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \cong \frac{\pi_1(U)}{\langle\langle \gamma \rangle\rangle},$$

where γ is the image in $\pi_1(U)$ of the generator of $\pi_1(U\cap V)$, and $\langle\langle\gamma\rangle\rangle$ denotes the smallest normal subgroup containing γ.

The homotopy  sends γ to $aba^{-1}b^{-1}$, and so

$$\pi_1(T) = \frac{\mathbb{Z}a * \mathbb{Z}b}{\langle\langle aba^{-1}b^{-1}\rangle\rangle} \cong \mathbb{Z}a \oplus \mathbb{Z}b$$

Following the same procedure,

$$\pi_1(K) = \frac{\mathbb{Z}a * \mathbb{Z}b}{\langle\langle abab^{-1}\rangle\rangle}$$

d) $\pi_1(T)$ is already abelian, and the abelianization of $\pi_1(K)$ is

$$\frac{\mathbb{Z}a + \mathbb{Z}b}{\langle a+b+a-b\rangle} \cong \frac{\mathbb{Z}a}{\langle 2a\rangle} \oplus \mathbb{Z}b$$

Problem 2

Let $\gamma \in \pi_1(X, x)$ be represented by $g: [0,1] \longrightarrow X$. Then $f_\#(\gamma)$ is represented by $f\circ g$. Therefore,

$$\phi_Y(f_\#(\gamma)) = [f\circ g] = f_*([g]) = f_*\,\phi_X(\gamma)$$

Problem 3:

If f, g are two homotopic maps, the following is a chain homotopy of complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_2(X) \arrow[r, "\partial_2"] \arrow[d, shift right, "g_2"'] \arrow[d, shift left, "f_2"] & C_1(X) \arrow[r, "\partial_1"] \arrow[dl, "P_1"'] \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "f_1"] & C_0(X) \arrow[r, "\varepsilon"] \arrow[dl, "P_0"'] \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "f_0"] & \mathbb{Z} \arrow[r] \arrow[dl, "0"'] \arrow[d, shift right, "id"'] \arrow[d, shift left, "id"] & 0 \arrow[r] \arrow[dl] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_2(Y) \arrow[r, "\partial_2"] & C_1(Y) \arrow[r, "\partial_1"] & C_0(Y) \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where P is the prism map defined in the lectures.

Problem 4:

Give the names j, q to the maps $A' \longrightarrow B'$ and $B \longrightarrow C$, respectively

Defining the maps:

- $\tilde{j}: \ker(f) \longrightarrow \ker(g): a' \longmapsto j(a')$. One needs to check that $f(a') = 0 \Rightarrow$
  $\Rightarrow 0 = i f(a') = g j(a') \Rightarrow j(a') \in \ker g$
- $\tilde{p}: \ker(g) \longrightarrow \ker(h): b' \longmapsto p(b')$. Same check
- $\partial: \ker(h) \longrightarrow \operatorname{coker}(f): c' \longmapsto a + \operatorname{im}(f)$ if $\exists\, b' \in B'$ s.t. $p(b') = c'$ and $g(b') = i(a)$. Existence of such an a: Since p is surjective, $\exists\, b'$ s.t. $p(b') = c'$, and since $h(c') = 0$, $g(b) \in \ker(q) = \operatorname{im}(i)$. It is well-defined because if there are a_1, a_2, b_1', b_2' such that $p(b_1') = p(b_2') = c'$ and $g(b_n') = i(a_n)$ for $n = 1, 2$, then $b_1' - b_2' \in \ker(j)$
  $\Rightarrow b_1' - b_2' = j(a') \Rightarrow i(a_1 - a_2) = i(f(a)) \Rightarrow a_1 - a_2 \in \operatorname{im}(f)$
  since i is injective.
- $\tilde{i}: \operatorname{coker}(f) \longrightarrow \operatorname{coker}(g): a + \operatorname{im}(f) \longmapsto i(a) + \operatorname{im}(g)$. One needs to check that if $a_1 - a_2 = f(a') \in \operatorname{im}(f)$ then $i(a_1) - i(a_2) = g(j(a')) \in \operatorname{im} g$
- $\tilde{q}: \operatorname{coker}(g) \longrightarrow \operatorname{coker}(h): b + \operatorname{im}(g) \longmapsto q(b) + \operatorname{im}(h)$. Same check

Exactness at $\ker(g)$: Let $b' \in \ker(g) \subseteq B'$. Then $\tilde{p}(b') = 0$ iff $\exists\, a' \in A'$ such that $j(a') = b'$, but any such a' satisfies $i(f(a')) = g(b') = 0$, so, since i is injective, such an a' is in $\ker(f)$.

Exactness at $\ker(h)$: Let $c' \in \ker(h)$. Then $\partial(c')=0$ iff there is some $b' \in B'$ such that $g(b)=i(0)=0$; i.e. iff $c'=p(b')$ for some $b' \in \ker(g)$.

Exactness at $\operatorname{coker}(f)$: Let $a+\operatorname{im}(f) \in \operatorname{coker}(f)$. Then $\tilde{\imath}(a)=0$ iff $i(a) \in \operatorname{im}(g)$ $\iff \exists\, b' \in B'$ such that $g(b')=i(a)$. Since for any such b', $h(p(b'))=q(g(b'))=q\,i(a)=0$, this is equivalent to $a+\operatorname{im}(f)$ being in the image of ∂.

Exactness at $\operatorname{coker}(g)$: Let $b+\operatorname{im}(g) \in \operatorname{coker}(g)$. Then $\tilde{q}(a+\operatorname{im}(g))=0$ iff $q(a) \in \operatorname{im}(h)$ $\iff \exists\, c' \in C$ such that $h(c')=q(a)$, but since p is surjective, this is equivalent to the existence of some $b' \in B'$ such that $q(b)=h(p(b'))=q(g(b'))$, and by exactness of the bottom row, we end up seeing that $\tilde{q}(b+\operatorname{im}(g))=0$ iff $\exists\, b' \in B' \mid b-g(b')=i(a)$ for some $a \in A$; i.e. iff $b+\operatorname{im}(g) \in \operatorname{im}(\tilde{\jmath})$.

Problem 5

If $C=0$, we can break the exact sequence into the smaller sequences

$$A \xrightarrow{f} B \longrightarrow 0 \quad , \quad 0 \longrightarrow D \xrightarrow{l} E$$

from which it follows that f is surjective and l is injective. For the converse, name the maps as follows:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{l} E$$

Then:

$$\left.\begin{array}{l} \to f \text{ surjective} \Rightarrow \operatorname{im} f = B \overset{\text{exactness at } B}{\Longrightarrow} \ker g = B \overset{g=0}{\Longrightarrow} \operatorname{im}(g)=0 \\ \to l \text{ injective} \Rightarrow \ker l = 0 \overset{\text{exactness at } D}{\Longrightarrow} \operatorname{im} h = 0 \overset{h=0}{\Longrightarrow} \ker(h)=C \end{array}\right\} \text{ by exactness at } C,\ C=0$$

If $A \subset X$, the long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_n} H_n(X) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A) \xrightarrow{i_{n-1}} H_{n-1}(X) \longrightarrow \cdots$$

tells us that $H_n(X,A)=0$ iff i_n is surjective and i_{n-1} is injective.

Therefore, $H_n(X,A)=0\ \forall n \iff i_n$ surjective and injective $\forall n \iff i_n$ isom. $\forall n$