Problem 1

If a SES $0 \longrightarrow \mathbb{Z}_{p^m} \longrightarrow A \longrightarrow \mathbb{Z}_{p^n} \longrightarrow 0$ exists, then A must be a finite group of cardinality p^{n+m} and generated by at most 2 elements. Moreover, we can see that it has an element of order at least $p^{\max\{n,m\}}$. By the classification of finite abelian groups, $A \cong \mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$, where $a \leq b$, $\max\{n,m\} \leq b$ and $a+b=n+m$. Now, let $\max\{n,m\} \leq b \leq n+m$. Note that this implies that $n+m-b \leq b$.

Claim: Under these conditions, a SES

$$0 \longrightarrow \mathbb{Z}_{p^m} \xrightarrow{f} \mathbb{Z}_{p^{n+m-b}} \times \mathbb{Z}_{p^b} \xrightarrow{g} \mathbb{Z}_{p^n} \longrightarrow 0$$

always exists.

Proof: Let $h : \mathbb{Z} \longrightarrow \mathbb{Z}_{p^{n+m-b}} \times \mathbb{Z}_{p^b}$ be the homomorphism sending 1 to $(1, p^{b-m})$. It is easy to see that $\ker(h) = (p^m)$ (Using that $b \geq \max\{n,m\}$) and so h induces an injective homomorphism $f: \mathbb{Z}_{p^m} \longrightarrow \mathbb{Z}_{p^{n+m-b}} \times \mathbb{Z}_{p^b}$ Since $(1, p^{b-m})$ and $(0,1)$ generate $\mathbb{Z}_{p^{n+m-b}} \times \mathbb{Z}_{p^b}$, the image of $(0,1)$ in $\operatorname{coker}(f)$ generates $\operatorname{coker}(f)$. Therefore, $\operatorname{coker}(f) \cong \mathbb{Z}_{p^n}$, since both are cyclic groups of the same cardinality. The claim follows because

$$0 \longrightarrow M \xrightarrow{f} N \longrightarrow \operatorname{coker}(f) \longrightarrow 0$$

is alway exact if f is injective.

If a SES $0 \longrightarrow \mathbb{Z} \xrightarrow{f} B \xrightarrow{g} \mathbb{Z}_n \longrightarrow 0$ exists, then B is an abelian group of rank 1, generated by at most 2 elements, so $B \cong \mathbb{Z} \oplus \mathbb{Z}_d$. Let $(a,b) = f(1)$, then the cokernel of the map $\phi : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ given by $\begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ is isomorphic to $\operatorname{coker}(f) = \mathbb{Z}_n$. It is well-known that $|\operatorname{coker}(\phi)| = |\det\begin{pmatrix} a & 0 \\ b & d \end{pmatrix}|$, so $d \mid n$. On the other hand, if $d \mid n$, the map $\mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}_d$ sending 1 to $(\frac{n}{d}, 1)$ is injective, and its cokernel is cyclic of order n ($(1,0)$ is a generator).

Problem 2

This follows from the reduced LES:

$$\cdots \longrightarrow \underset{\overset{\shortparallel}{0}}{\tilde{H}_n(A)} \longrightarrow \tilde{H}_n(X) \underset{\overset{\uparrow}{\text{isom.}}}{\longrightarrow} \tilde{H}_n(X,A) \longrightarrow \underset{\overset{\shortparallel}{0}}{\tilde{H}_{n-1}(A)} \longrightarrow \cdots$$

Problem 3

By looking at the LES

$$H_1(A) \longrightarrow H_1(X) \longrightarrow H_1(X,A) \longrightarrow H_0(A) \longrightarrow H_0(X) \longrightarrow H_0(X,A) \longrightarrow 0$$

we see that $H_0(X,A)=0$ iff $H_0(A) \longrightarrow H_0(X)$ is surjective and that $H_1(X,A)=0$ iff $H_1(A) \longrightarrow H_1(X)$ is surjective and $H_0(A) \longrightarrow H_0(X)$ is injective.

So we need to analize what is the map $H_0(A) \longrightarrow H_0(X)$. Recall that for any space Y, $H_0(Y)$ is generated by the set of path components of Y, and so, the map $H_0(A) \longrightarrow H_0(X)$ is just

$$\bigoplus_{\substack{\text{path}\\\text{components}\\\text{of } A}} \mathbb{Z} \longrightarrow \bigoplus_{\substack{\text{path}\\\text{components}\\\text{of } X}} \mathbb{Z} \ : \ [p] \longmapsto [p]$$

and the two claims a), b) follow from this description.

Problem 4

Consider the LES

$$H_1(\mathbb{Q}) \longrightarrow \underset{\overset{\shortparallel}{0}}{H_1(\mathbb{R})} \longrightarrow H_1(\mathbb{R},\mathbb{Q}) \longrightarrow \tilde{H}_0(\mathbb{Q}) \longrightarrow \underset{\overset{\shortparallel}{0}}{\tilde{H}_0(\mathbb{R})} \longrightarrow \tilde{H}_0(\mathbb{R},\mathbb{Q})$$

Therefore $H_1(\mathbb{R},\mathbb{Q}) \cong \tilde{H}_0(\mathbb{Q}) = \bigoplus_{p\in\mathbb{Q}\setminus\{0\}} \mathbb{Z}\langle [p]-[0] \rangle$, which has a countable basis

Problem 5

a) Follows from the commutativity of

$$\begin{tikzcd}
(X, A) \arrow[r] \arrow[d] & (X, V) \arrow[d] & (X \setminus A, V \setminus A) \arrow[l] \arrow[d] \\
(X/A, A/A) \arrow[r] & (X/A, V/A) & (X/A - A/A, V/A - A/A) \arrow[l]
\end{tikzcd}$$

and the naturality of homology groups.

b) · $\tilde{H}_p(X, A) \longrightarrow \tilde{H}_p(X, V)$ is an isomorphism because $\tilde{H}_i(A, V) = 0$ for all i, since $A \subset V$ is a deformation retract, and also using the LES of the triple (A, V, X).

$\tilde{H}_p(X \setminus A, V \setminus A) \longrightarrow \tilde{H}_p(X, V)$ is an isomaphism due to excision.

· $H_p(X/A, A/A) \longrightarrow H_p(X/A, V/A)$ is an isomorphism because, again, $\tilde{H}_i(V/A, A/A) = 0$ $\forall i$ because $A \subseteq V$ is a <u>strong</u> deformation retract

↳ important !!

and again a LES of a triple.

· $\tilde{H}_p(X/A \setminus A/A, V/A \setminus A/A) \longrightarrow H_p(X/A, V/A)$ is an iso due to excision.

c) This is because $q : (X \setminus A, V \setminus A) \longrightarrow (X/A \setminus A/A, V/A \setminus A/A)$ is a homeomorphism of pairs.

d) Recall that for a nonempty space X, $\tilde{H}_p(X) \cong \tilde{H}_p(X, \{pt\})$. Using the commutativity of the diagram and that everything besides the middle and left vertical arrows are an isomorphism, one checks that $\tilde{H}_p(X, A) \xrightarrow{q_*} \tilde{H}_p(X/A, A/A) \cong \tilde{H}_p(X/A)$ is an isomorphism.

e) We know that there is a diagram with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_n(A) \arrow[r, "i_*"] \arrow[d] & \tilde{H}_n(X) \arrow[r] \arrow[d, "j_*"] & \tilde{H}_n(X, A) \arrow[r, "\partial"] \arrow[d, "q_*", "\cong"'] & \tilde{H}_{n-1}(A) \arrow[r, "i_*"] \arrow[d] & \cdots \\
\cdots \arrow[r] & \tilde{H}_n(A/A) \arrow[r] \arrow[d, equal] & \tilde{H}_n(X/A) \arrow[r, "\cong", "\delta"'] & \tilde{H}_n(X/A, A/A) \arrow[r] & \tilde{H}_{n-1}(A/A) \arrow[r] \arrow[d, equal] & \cdots \\
& 0 & & & 0 &
\end{tikzcd}$$

so we can form the desired LES by letting $\tilde{H}_n(X/A) \longrightarrow \tilde{H}_n(A)$ be $\partial \circ (q_*^{-1}) \circ \delta$

Problem 6:

Let $A=\{[t,x]\in\Sigma X : t\geq 1/4\}$, $B=\{[t,x]\in\Sigma X \mid t\leq 3/4\}$. A and B are contractible, and $\{1/2\}\times X\subseteq A\cap B$ is a strong deformation retract. Since $\text{int}(A)\cup\text{int}(B)=\Sigma X$, we can apply Mayer-Vietoris:

$$\begin{tikzcd}
\widetilde{H}_{n+1}(A) \arrow[r, phantom, "\oplus"] \arrow[d, equal] & \widetilde{H}_{n+1}(B) \arrow[r] \arrow[d, equal] & \widetilde{H}_{n+1}(\Sigma X) \arrow[r] & \widehat{H}_n(A\cap B) \arrow[r] & \widetilde{H}_n(A) \arrow[r, phantom, "\oplus"] \arrow[d, equal] & \widetilde{H}_n(B) \arrow[d, equal] \\
0 & 0 & & \widetilde{H}_n(X) \arrow[u, "\cong"'] & 0 & 0
\end{tikzcd}$$

Naturality follows because

$$\begin{tikzcd}
\{1/2\}\times X \arrow[r] \arrow[d, "f"] & \Sigma X \arrow[d, "\Sigma f"] \\
\{1/2\}\times Y \arrow[r] & \Sigma Y
\end{tikzcd}$$

commutes.